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Health Care and Economic Growth

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ABSTRACT. – In this paper we adapt a discrete time version of the Lucas model to a model with social protection where part of the total production is devoted to the health expenditures. The output is produced by labor and the technology exhibits externalities. The rate of growth of human capital depends on the ratio of health expenditures over GDP. We give conditions for which the optimal human capital sequences are increasing. When the instantaneous utility function is isoelastic and the production function is COBB-DOUGLAS, we prove that the optimal human capital sequences grow at constant rate. Moreover, we prove there exists a unique equilibrium in the sense of LUCAS [1988] or ROMER [1986].

Soins de santé et croissance économique

RÉSUMÉ. – Dans ce papier, nous adaptons une version en temps discret du modèle de Lucas de façon à prendre en compte la protection sociale, financée par une partie de la production. Le travail est le seul facteur de production et la technologie est caractérisée par des externalités. Le taux de croissance du capital humain dépend du ratio des dépenses publiques dans le PIB. Nous donnons les conditions sous lesquelles la trajectoire du capital humain est croissante. Quand la fonction d'utilité indirecte est isoélastique et la fonction de production est Cobb-Douglas, nous montrons que le capital humain croît à taux constant. De plus, nous montrons qu'il existe un solution unique au sens de LUCAS [1988] et ROMER [1986].

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1 Introduction

The role of knowledge or human capital has been proved to be crucial for the endogenous growth theory. In ROMER [1986] knowledge accumulated by the agents is the basic form of the capital. In LUCAS [1988], physical capital and human capital are used as inputs in production process. But the crucial role is the human capital accumulation. For that reason, in this paper, we focus on the human capital. But we assume that its rate of growth depends on the ratio of health expenditures over total output. We consider a discrete time version model *à la* LUCAS where the human capital influences the production as input and as externality. We do not introduce physical capital in this model. Healthier workers are more productive because they are more physically and mentally robust. Although GARY BECKER defined human capital as investment on education, health and migration, most of empirical studies limit its definition to education investment. Hence, the provision for education requires resources and there seems to be a trade-off too between education and health expenditures. But the complementarity between health and education can be underlined (ZON AND MUYSKEN [2000]). Increases in health investment lengthen ones life span, and hence increases the return in investment education (FUCHS [1982]). On the other hand, a higher level of education may increase preference for health. This paper is limited to the study of impact of health expenditures on growth. For the influence of education and training on the human capital in a discrete time Lucas model, one can refer to GOURDEL, HOANG-NGOC, LE VAN and MAZAMBA [2003].

We first consider the social planner problem. As in BARRO [1990], we try to estimate the impact on economic growth of the public expenditures devoted to health cares. Actually, we show that if the quality and/or the externality effect of the human capital accumulation are high, then the economy will take off, *i.e.*, the optimal human capital sequence will grow over time. If we assume that the instantaneous utility function is isoelastic and the production function is COBB-DOUGLAS, then there exists a unique solution to the social planner problem, which grows at constant rate. This one positively depends on the externality parameter of the production function and the quality of the human capital technology.

Second, we show that, in our model, there exists a unique equilibrium. Equilibrium must be understood in the sense of LUCAS [1988] or ROMER, [1986]. That is a human capital path such that, when it is used as externality, it will coincide with the solution to the optimal problem taking it as exogenously determined.

The paper is organized as follows: in Section 2, we present the model and prove existence of the optimal solution to the social planner problem. In Section 3, we prove existence and uniqueness of equilibria. Section 4 is devoted to some concluding remarks.

2 The Social Planner Problem

2.1 The Model

We consider an intertemporal model where the social planner maximizes the utility of an infinitely lived representative consumer. The consumption good is produced through a production function using only labour as input. Effective labour is the sum of working hours combined with the human capital which are devoted to the production process. More explicitly, we assume there exists a representative worker who has $h \in [0, +\infty[$ as skill level. Effective labour is $N^e = h$. Given h, the production level is $G(h)f(h)$. The term $G(h)$ captures the external effect of the human capital. The rate of growth of the human capital depends on the expenses for health cares S_t. We assume that $S_t = \sigma_t G(h_t) f(h_t)$, with $0 \leqslant \sigma_t \leqslant 1$. The model is as follows :

$$\max \sum_{t=0}^{+\infty} \beta^t u(c_t)$$

under the constraints :

$$\forall t \geqslant 0, 0 \leqslant c_t \leqslant G(h_t) f(h_t)(1 - \sigma_t),$$

$$h_{t+1} = h_t(1 + \lambda\phi(\sigma_t)),$$

$$0 \leqslant \sigma_t \leqslant 1, \;\; h_0 > 0 \text{ is given.}$$

In the equation describing the dynamics of h_t, the parameter λ measures the quality of the human capital technology function ϕ.

We make the following assumptions :

$H1$: The utility function u is strictly concave, strictly increasing, continuously differentiable and satisfies INADA condition $u'(0) = +\infty$.

$H2$: The production function f is COBB-DOUGLAS : $f(x) = x^{\alpha}$, $0 < \alpha \leqslant 1$.

$H3$: The function G is of the form: $G(x) = x^{\gamma}$, with $\gamma \geqslant 0$.

$H4$: The function ϕ is strictly increasing and twice continuously differentiable, $\phi(0) = 0$, $\phi(1) = 1$, $\lambda > 0$.

$H5$: $0 < \beta(1 + \lambda)^{\alpha+\gamma} < 1$.

$H6$: $\alpha + \gamma \geqslant 1$.

In Assumption $H3$, the parameter γ is a measure of the magnitude of the externality effect of the human capital. With Assumption $H4$, λ is the maximum rate of growth of the human capital. Assumption $H5$ ensures the intertemporal utility $\sum_{t=0}^{+\infty} \beta^t u(c_t)$ to be well-defined. With Assumption $H6$ the production function of the social planner exhibits increasing returns to scale.

2.2 Optimal Solutions

The following proposition claims existence of optimal solutions.

PROPOSITION 1: *Under assumptions $H1 - H5$, there exists a solution.*

PROOF: It is quite standard. ■

Let $\psi : [1, 1+\lambda] \to \mathbb{R}$ be defined by

$$\psi(x) = 1 - \phi^{-1}\left(\frac{1}{\lambda}(x-1)\right)$$

where ϕ^{-1} denotes the inverse map of ϕ. The function ψ is clearly decreasing. It is easy to check that : $\psi(1) = 1$ and $\psi(1+\lambda) = 0$. The function ψ gives the working time when the human capital rate of growth is x.

We list the properties of ψ.

(a) ψ is continuously differentiable, decreasing,

$$\psi(1) = 1,\ \psi(1+\lambda) = 0,\ \psi'(1) = -\frac{1}{\lambda\phi'(0)},\ \psi'(1+\lambda) = -\frac{1}{\lambda\phi'(1)}$$

(b) If ϕ is (strictly) concave, then ψ is also (strictly) concave.

Observe that the problem is now equivalent to :

$$\max \sum_{t=0}^{+\infty} \beta^t u\left(G(h_t) f(h_t) \psi\left(\frac{h_t}{h_{t+1}}\right)\right)$$

under the constraints :

$$\forall t \geqslant 0,\ h_t \leqslant h_{t+1} \leqslant h_{t+1}(1+\lambda) \text{ and } h_0 > 0 \text{ is given.}$$

The following proposition states that under an additional assumption the optimal sequence of human capital is strictly increasing. In other words, the economy will take off, *i.e.*, will not stick at the initial value $h_0 > 0$.

PROPOSITION 2: *Assume $H1 - \ldots - H5$ and*

H7 : $$\lambda\phi'(0) > \left(\frac{1}{\beta} - 1\right)\frac{1}{\alpha+\gamma}.$$

Then any optimal human capital sequence $\mathbf{h} = (h_0, h_1, \ldots, h_t, \ldots)$ *satisfies* $h_0 < h_1 < \ldots < h_t < \ldots$.

PROOF: Since the problem is stationary, it suffices to show that for any $h_0 > 0$, the stationary sequence $(h_0, h_0, \ldots, h_0, \ldots)$ is not optimal.

Let $\varepsilon > 0$ sufficiently small such that $(1 + \lambda\phi(\varepsilon)) \leqslant (1 + \lambda)$. Define the sequence $\mathbf{h} = (h_0, h_1, .., h_t, ...)$ by $h_t = h_0(1 + \lambda\phi(\varepsilon))$ for any $t \geqslant 1$. The associated sequence of consumptions $\mathbf{c}_\varepsilon = (c_{0\varepsilon}, c_{1\varepsilon}, ..., c_{t\varepsilon}, ...)$ is $c_{0\varepsilon} = G(h_0)f(h_0)(1 - \varepsilon)$ and $c_{t\varepsilon} = G(h_0(1 + \lambda\phi(\varepsilon))f(h_0(1 + \lambda\phi(\varepsilon))$ for any $t \geqslant 1$.

The sequence of consumptions $\mathbf{c}^*$ associated with $(h_0, h_0, ..., h_0, ...)$ is $c_t^* = G(h_0)f(h_0)$ for every t. We compare the utilities associated with these sequences of consumptions. Let

$$\Delta_\varepsilon = \sum_{t=0}^{+\infty} \beta^t u(c_t) - \sum_{t=0}^{+\infty} \beta^t u(c_t^*).$$

From the concavity of u, one gets:

$$\begin{aligned} \Delta_\varepsilon \quad & \geqslant u'(c_{0\varepsilon})[-\varepsilon G(h_0)f(h_0)] \\ & + \frac{\beta}{1-\beta} u'(c_{1\varepsilon}) \left(G(h_0)(1 + \lambda\phi(\varepsilon))f(h_0(1 + \lambda\phi(\varepsilon)) - G(h_0)f(h_0) \right). \end{aligned}$$

It is easy to show that

$$\lim_{\varepsilon \to 0} \frac{\Delta_\varepsilon}{\varepsilon} \geqslant u'(c_0^*) h_0^{\alpha+\gamma} \left[-1 + \frac{\beta\lambda}{1-\beta}(\alpha + \gamma)\phi'(0) \right].$$

Therefore, if $\lambda\phi'(0) > \dfrac{1}{\alpha+\gamma} \dfrac{1-\beta}{\beta}$, then $\Delta_\varepsilon > 0$ for ε small enough. In other words, the stationary sequence $(h_0, h_0, ..., h_0, ...)$ is not optimal. ■

We now add assumptions in order to obtain uniqueness of optimal human capital paths which grow at constant rate:

$H8$: The utility function u has the form $u(c) = c^\mu$ with $0 < \mu < 1$. Moreover: $(\alpha + \gamma)\mu - 1 < 0$.

$H9$: The function ϕ is concave.

Assumption $H8$ restricts the magnitude of the externality parameter γ. From $H9$ the human capital technology has decreasing returns to scale.

Remark. Assumptions $H4$ and $H9$ imply $\phi'(0) \geqslant 1$ and $\phi'(0) = 1$ if and only if $\phi(x) = x$, for all x.

In the following proposition, aside the result on the uniqueness of optimal human capital paths with constant growth rate, we show that assumption $H7$ is also necessary for the economy to take off. We also show that the constant growth rate of optimal human capital increases with the externality parameter and the quality of the human capital technology.

PROPOSITION 3:

1. Assume $H1 - H2 - \ldots - H9$. Then, the optimal human capital sequence is unique and grows at constant rate $\nu \in]1,1+\lambda[$. This rate ν increases with λ and γ.

2. Assume $H1 - H2 - H3 - H4 - H5 - H6 - H8 - H9$. If $H7$ does not hold, then the optimal human capital path is the stationary sequence $(h_0, h_0, \ldots, h_0, \ldots)$.

PROOF: The proof is done in two steps.

Step 1. Let V denote the value function, *i.e.*

$$V(h_0) = \max \sum_{t=0}^{+\infty} \beta^t h_t^{(\alpha+\gamma)\mu} \left[\psi\left(\frac{h_{t+1}}{h_t}\right)\right]^\mu$$

under the constraints:

$$\forall t \geqslant 0,\ h_t \leqslant h_{t+1} \leqslant h_t(1+\lambda), \text{ and } h_0 > 0 \text{ is given.}$$

(a) The value function has the form $V(h_0) = Ah_0^{(\alpha+\gamma)\mu}$ (see *e.g.* LE VAN and MORHAIM [2002]).

(b) Given h_0, the optimal value of the human capital of period 1 is $h_1^* = \nu h_0$ with ν solution to

$$\max \left\{[\psi(z)]^\mu + \beta A z^{(\alpha+\gamma)\mu} : z \in [1,(1+\lambda)]\right\}$$

(see LE VAN and DANA [2003]).

Therefore, if $\{h_t\}$ is an optimal sequence, we have $h_t = \nu^t h_0$ for every t.

Step 2. From proposition 2, an optimal sequence of human capital must satisfy $h_{t+1} > h_t$ for any $t \geqslant 0$. Since $u'(0) = +\infty$, optimal consumptions must be positive. Thus EULER equation holds: $\forall t$,

$$h_t^{(\alpha+\gamma)\mu-1}\left(\psi\left(\frac{h_{t+1}}{h_t}\right)\right)^{\mu-1}\psi'\left(\frac{h_{t+1}}{h_t}\right) + (\alpha+\gamma)\beta\left(h_{t+1}\right)^{(\alpha+\gamma)\mu-1}\left(\psi\left(\frac{h_{t+2}}{h_{t+1}}\right)\right)^\mu$$
$$-\beta h_{t+1}^{(\alpha+\gamma)\mu-1}\frac{h_{t+2}}{h_{t+1}}\left(\psi\left(\frac{h_{t+2}}{h_{t+1}}\right)\right)^{\mu-1}\psi'\left(\frac{h_{t+2}}{h_{t+1}}\right) = 0.$$

From Step 1, we know that the optimal sequence of human capital has constant growth rate ν. From Euler equation, we get the following equation for ν:

$$\nu^{1-(\alpha+\gamma)\mu} = \beta\left[-(\alpha+\gamma)\frac{\psi(\nu)}{\psi'(\nu)} + \nu\right]. \quad (1)$$

Let $H_\gamma(\nu) = \nu^{1-(\alpha+\gamma)\mu}, F_{\gamma,\lambda}(\nu) = \beta\left[-(\alpha+\gamma)\frac{\psi(\nu)}{\psi'(\nu)} + \nu\right]$. The function H_γ is obviously increasing. For the second function, we have $F'_{\gamma,\lambda}(x) = \beta[1-\alpha-\gamma+(\alpha+\gamma)\frac{\psi(x)}{(\psi'(x))^2}\psi''(x)] < 0$. Since $F_{\gamma,\lambda}(1) = \beta[(\alpha+\gamma)\lambda\phi'(0)+1] > 1 = H_\gamma(1)$ by $H7$, and $F_{\gamma,\lambda}(1+\lambda) = \beta(1+\lambda) < H_\gamma(1+\lambda) = (1+\lambda)^{1-(\alpha+\gamma)\mu}$ because $\mu < 1$ and $H5$ holds. There thus exists a unique solution $\nu \in]1,1+\lambda[$. We have proved that the optimal human capital sequence is unique and grows at constant rate.

We now show that this unique solution ν increases with γ and λ. Observe that $\frac{\partial H_\gamma}{\partial \gamma} = -\mu H_\gamma(\nu)Log(\nu)$, $\frac{\partial F_{\gamma,\lambda}}{\partial \gamma} = -\beta\frac{\psi(\nu)}{\psi'(\nu)}$.

We now prove that $\frac{\partial F_{\gamma,\lambda}}{\partial \lambda} > 0$. Recall that $\psi_\lambda(x) = 1-\phi^{-1}(\frac{1}{\lambda}(x-1))$. Since ϕ^{-1} is increasing, ψ is then increasing in λ. It is easy to find that $\psi'(x) = -\frac{1}{\lambda\phi'(\sigma)}$ with $\sigma = \phi^{-1}\left(\frac{1}{\lambda}(x-1)\right)$. Hence, when λ increases, then σ decreases. Consequently, since ϕ is concave, $\lambda\phi'(\sigma)$ increases and hence, ψ' increases in λ. After tedious computation, we prove that $\frac{\partial F_{\gamma,\lambda}}{\partial \lambda} > 0$.

Differentiating equation (1), we obtain:

$$\frac{\partial \nu}{\partial \gamma} = \frac{\mu Log(\nu)H_\gamma(\nu) - \beta\frac{\psi(\nu)}{\psi'(\nu)}}{H'_\gamma(\nu) - F'_{\gamma,\lambda}(\nu)} > 0,$$

and

$$\frac{\partial \nu}{\partial \lambda} = \frac{\partial F_{\gamma,\lambda}}{\partial \lambda}\frac{1}{H'_\gamma(\nu) - F'_{\gamma,\lambda}(\nu)} > 0.$$

2. We know that there exists a solution which grows at constant rate ν. If it is interior, then ν is determined by the intersection of the graphs of H_γ which is increasing and of $F_{\gamma,\lambda}$ which is decreasing. Assume that $H7$ is not satisfied. The graphs of H_γ and $F_{\gamma,\lambda}$ have no intersection in the interval $[1,1+\lambda]$ if $\lambda\phi'(0) < \left(\frac{1}{\beta}-1\right)\frac{1}{\alpha+\gamma}$ or intersect only at the point $(1,1)$ if $\lambda\phi'(0) = \left(\frac{1}{\beta}-1\right)\frac{1}{\alpha+\gamma}$. We conclude that the optimal path can not be interior and ν equals 1 or $1+\lambda$. But ν must differ from $1+\lambda$, since, in this case, the optimal consumptions equal zero for every period. That is impossible, since $h_0 > 0$. Hence, the optimal path must be the stationary sequence $(h_0,\ldots,h_0,\ldots)$. ■

3 Existence and Uniqueness of Equilibrium

We first define the concepts of equilibrium (in the sense of LUCAS or ROMER) and competitive equilibrium.

Suppose we are given a sequence of human capital $\overline{\mathbf{h}} = (h_0, \overline{h}_1, ..., \overline{h}_t, ...)$. Consider the following model:

$$\max \sum_{t=0}^{t=+\infty} \beta^t u(c_t)$$

under the constraints:

$$\text{for any } t,\ 0 \leqslant c_t \leqslant G(\overline{h}_t) f(h_t)(1-\sigma_t),$$

$$h_{t+1} = h_t(1+\lambda\phi(\sigma_t)),$$

$$0 \leqslant \sigma_t \leqslant 1,\ \ h_0 > 0 \text{ is given.}$$

The solution $\mathbf{h} = (h_0, \ldots h_t, \ldots)$ to this problem depends on $\overline{\mathbf{h}}$. We write $\mathbf{h} = \Phi(\overline{\mathbf{h}})$. An *equilibrium* is a sequence of human capital $\mathbf{h}^* = (h_0, h_1^*, ..., h_t^*, \ldots)$ such that $\mathbf{h}^* = \Phi(\mathbf{h}^*)$.

We give below conditions for which an equilibrium $\mathbf{h}^*$ is strictly increasing.

PROPOSITION 4: *Assume $H1 - H5$ and*

$H7b$: $\lambda\phi'(0) > \left(\frac{1}{\beta} - 1\right)\frac{1}{\alpha}$.

Then, any equilibrium $\mathbf{h}^*$ is strictly increasing.

PROOF: Assume the contrary. We have two cases.

Case 1. The optimal sequence $\mathbf{h}^*$ satisfies $h_t^* = h_T^*$ for any $t \geqslant T$. Let ε satisfy $0 < \varepsilon < 1 + \lambda$. Define a sequence $\mathbf{h}$ by $h_t = h_t^*, \forall t \leqslant T$ and $h_t = h_T^* + \varepsilon$, for $t > T$. We will show that, with $\mathbf{h}^*$ as externality, the intertemporal utility generated by $\mathbf{h}$ is greater than the one generated by $\mathbf{h}^*$, which contradicts the optimality of $\mathbf{h}^*$.

Let

$$\Delta_\varepsilon = \sum_{t=0}^{+\infty} \beta^t u\left(G(h_t^*) f(h_t)\psi\left(\frac{h_{t+1}}{h_t}\right)\right) - \sum_{t=0}^{+\infty} \beta^t u\left(G(h_t^*) f(h_t^*)\psi\left(\frac{h_{t+1}^*}{h_t^*}\right)\right).$$

One gets from the concavity of u, and f:

$$\Delta_\varepsilon \geqslant \beta^T u'\left(G(h_T^*)f(h_T^*)\psi\left(\frac{h_T^*+\varepsilon}{h_T^*}\right)\right)G(h_T^*)f(h_T^*)\left[\psi\left(\frac{h_T^*+\varepsilon}{h_T^*}\right)-\psi(1)\right]$$
$$+\sum_{t\geqslant T+1}\beta^t u'(G(h_T^*)f(h_T^*+\varepsilon)\psi(1))G(h_T^*)f'(h_T^*+\varepsilon)\varepsilon.$$

Thus:

$$\lim_{\varepsilon\to 0}\frac{\Delta_\varepsilon}{\varepsilon} \geqslant \beta^T u'(G(h_T^*)f(h_T^*))G(h_T^*)\left[\frac{\beta}{1-\beta}f'(h_T^*)-\frac{1}{\lambda\phi'(0)}\frac{f(h_T^*)}{h_T^*}\right].$$

Replace $f(x)=x^\alpha$. We obtain

$$\left[\frac{\beta}{1-\beta}f'(h_T^*)-\frac{1}{\lambda\phi'(0)}\frac{f(h_T^*)}{h_T^*}\right]=(h_T^*)^{\alpha-1}\left[\frac{\beta}{1-\beta}\alpha-\frac{1}{\lambda\phi'(0)}\right]>0\,.$$

Hence $\Delta_\varepsilon>0$ for $\varepsilon>0$ sufficiently small.

Case 2. The optimal sequence $\mathbf{h}^*$ satisfies $h_t^*=h_T^*$ for $T\leqslant t\leqslant T+\tau$. Define $\mathbf{h}$ by $h_t=h_t^*, \forall t\leqslant T+\tau-1$,

$$h_{T+\tau}^*<h_{T+\tau}=h_{T+\tau}^*+\varepsilon<h_{T+\tau+1}^*,$$

and $h_{T+t}=h_{T+t}^*$ for $t\geqslant\tau+1$. As previously, we will show that, with $\mathbf{h}^*$ as externality, the intertemporal utility generated by $\mathbf{h}$ is greater than the one generated by $\mathbf{h}^*$. Let

$$\Delta_\varepsilon=\sum_{t=0}^{\infty}\beta^t u\left(G(h_t^*)f\left(h_t\psi\left(\frac{h_{t+1}}{h_t}\right)\right)\right)$$
$$-\sum_{t=0}^{\infty}\beta^t u\left(G(h_t^*)f\left(h_t^*\psi\left(\frac{h_{t+1}^*}{h_t^*}\right)\right)\right).$$

The technics used above again gives $\Delta_\varepsilon>0$ for $\varepsilon>0$ sufficiently small. ■

The following proposition gives necessary and sufficient conditions for a sequence $\mathbf{h}^*$ to be an equilibrium.

PROPOSITION 5: *Assume* $H1-H2-\ldots-H5-H6-H7b-H9$ *and* $H10$: *The function* $(x,y)\to x^\alpha\psi\left(\frac{y}{x}\right)$ *is concave when* $x\leqslant y\leqslant(1+\lambda)x$.

A sequence $\mathbf{h}^*$ *is an equilibrium starting from* $h_0>0$ *if, and only if, it satisfies the following conditions:*

(1). Interiority:

$$\forall t\geqslant 0,\ h_t^*<h_{t+1}^*<(1+\lambda)\,h_t^*,\ h_0^*=h_0>0,$$

(2). Euler equation: $\forall t \geqslant 0$,

$$-u'(c_t^*)\left(h_t^*\right)^{\alpha+\gamma-1}\psi'\left(\frac{h_{t+1}^*}{h_t^*}\right)$$

$$= \beta u'(c_{t+1}^*)\left(h_{t+1}^*\right)^{\alpha+\gamma-1}\left[\alpha\psi\left(\frac{h_{t+2}^*}{h_{t+1}^*}\right) - \frac{h_{t+2}^*}{h_{t+1}^*}\psi'\left(\frac{h_{t+2}^*}{h_{t+1}^*}\right)\right],$$

(3). Transversality condition:

$$\lim_{t\to\infty}\beta^t u'(c_t^*)\left(h_t^*\right)^{\alpha+\gamma-1}\psi'\left(\frac{h_{t+1}^*}{h_t^*}\right)h_{t+1}^* = 0.$$

PROOF: 1. Let $\mathbf{h}^*$ be an equilibrium. From the previous proposition, we have $h_{t+1}^* > h_t^*, \forall t \geqslant 0$. Since u satisfies Inada condition $u'(0) = +\infty$, optimal consumptions must be positive at each period. Hence, $h_{t+1}^* < (1+\lambda)h_t^*$, for every t.

Since the optimal path $\mathbf{h}^*$ is interior, EULER equation must hold (see *e.g.* LE VAN and DANA [2003]).

We now prove that the transversality condition also holds.

Let

$$V_{\mathbf{h}^*}(h_0) = \max\sum_{t=0}^{\infty}\beta^t u\left(\left(h_t^*\right)^{\gamma} h_t^{\alpha}\psi\left(\frac{h_{t+1}}{h_t}\right)\right)$$

under the constraints

$$\forall t \geqslant 0,\ h_t \leqslant h_{t+1} \leqslant (1+\lambda)h_t,\ \text{and } h_0 \text{ is given.}$$

The function $V_{\mathbf{h}^*}$ is concave.

Let S denote the shift operator, *i.e.* $S\mathbf{h}^* = (h_1^*, h_2^*, \ldots)$, $\forall T$, $S^T\mathbf{h}^* = (h_T^*, h_{t+1}^*, \ldots)$. From BENVENISTE and SCHEINKMAN [1979], $V_{\mathbf{h}^*}$ is differentiable and

$$\forall t,\ V'_{S^t\mathbf{h}^*}(h_t) = u'(c_t)h_t^{*\alpha+\gamma-1}\left[\alpha\psi\left(\frac{h_{t+1}}{h_t}\right) - \frac{h_{t+1}}{h_t}\psi'\left(\frac{h_{t+1}}{h_t}\right)\right]$$

where $(h_1, \ldots, h_t, \ldots)$ is the optimal sequence from h_0, and c_t is the associated consumption. Since $V_{\mathbf{h}^*}$ is concave we have:

$$\begin{aligned}
V_{S^t\mathbf{h}^*}(h_t^*) &= V_{S^t\mathbf{h}^*}(h_t^*) - V_{S^t\mathbf{h}^*}(0) \geqslant V'_{S^t\mathbf{h}^*}(h_t^*)h_t^* \\
&\geqslant u'(c_t^*)h_t^{*\alpha+\gamma-1}\left[\alpha\psi\left(\frac{h_{t+1}}{h_t}\right) - \frac{h_{t+1}}{h_t}\psi'\left(\frac{h_{t+1}^*}{h_t^*}\right)\right]h_t^* \\
&\geqslant -u'(c_t^*)h_t^{*\alpha+\gamma-1}\frac{h_{t+1}^*}{h_t^*}\psi'\left(\frac{h_{t+1}^*}{h_t^*}\right)h_t^*.
\end{aligned}$$

Since $\beta^t V_{S'\mathbf{h}^*}(h_t^*) \to 0$ when $t \to \infty$, we have

$$\lim_{t\to\infty} \beta^t u'(c_t^*) h_t^{*\alpha+\gamma-1} \frac{h_{t+1}^*}{h_t^*} \psi'\left(\frac{h_{t+1}^*}{h_t^*}\right) h_t^* = 0.$$

2. The proof that these conditions are sufficient is standard. ■

PROPOSITION 6: *Assume* $H1 - H2 - H3 - H4 - H5 - H6 - H9 - H10$, *and*

$H7b$: $\lambda\phi'(0) > \left(\frac{1}{\beta} - 1\right)\frac{1}{\alpha}$

$H8b$: $u(c) = c^{\mu}$ *with* $\mu \in]0,1[$, *and* $(\alpha + \gamma)\,\mu - \alpha < 0$.

Then there exists a unique equilibrium $\mathbf{h}^*$. *It grows at constant rate* ν. *This rate is smaller than the one in the social planner problem.*

PROOF: The strategy of proof is to show that

(a) the Euler equation admits a solution $\mathbf{h}^*$ which grows at constant rate $\nu^* \in]1, 1 + \lambda[$. Moreover, this solution satisfies the three conditions of the previous proposition and thus, is optimal,

(b) any other solution $\mathbf{h}$ to EULER equation does not satisfy the transversality condition. Hence, again from the previous proposition, it is not optimal. From that, one concludes that there exists a unique equilibrium.

We will show that there exists a solution $\mathbf{h}^*$ to EULER equation which grows at constant rate ν^*. Indeed, from EULER equation, ν^* must solve the following equation:

$$-\frac{\psi'(\nu)\nu^{1-(\alpha+\gamma)\mu}}{\alpha\psi(\nu) - \nu\psi'(\nu)} = \beta.$$

Let $H(\nu) = -\frac{\psi'(\nu)\nu^{1-(\alpha+\gamma)\mu}}{\alpha\psi(\nu) - \nu\psi'(\nu)}$. We obtain

$$H'(\nu) = A + B$$

where

$$A = \frac{-\alpha\nu^{1-(\alpha+\gamma)\mu}\psi''(\nu)\psi(\nu) - \alpha(1 - (\alpha+\gamma)\mu)\psi(\nu)\psi'(\nu)\nu^{-(\alpha+\gamma)\mu}}{\left(\alpha\psi(\nu) - \nu\psi'(\nu)\right)^2} > 0,$$

and

$$B = \frac{(\alpha - (\alpha+\gamma)\,\mu)\psi'(\nu)^2\nu^{1-(\alpha+\gamma)\mu}}{(\alpha\psi(\nu) - \nu\psi'(\nu))^2} > 0$$

Hence, if $(\alpha+\gamma)\mu-\alpha<0$, then $H'(\nu)>0$. We have $H(1)=\dfrac{1}{\lambda\phi'(0)+1}<\beta$, from Assumption $H7b$, while $H(1+\lambda)=(1+\lambda)^{-(\alpha+\gamma)\mu}<\beta$ from $H5$. Therefore, there exists a unique solution ν^*, which is in the interval $]1,1+\lambda[$. It is easy to show that this rate is smaller than the one in the social planner problem which solves $L(\nu)=\beta$ with $L(\nu)=-\dfrac{\psi'(\nu)\nu^{1-(\alpha+\gamma)\mu}}{(\alpha+\gamma)\psi(\nu)-\nu\psi'(\nu)}$ which is increasing and smaller than $H(\nu)$.

Let $\mathbf{h}^*$ be defined by $h_0^*=h_0$, $h_{t+1}^*=\nu^*h_t^*,\forall t$. Obviously, it satisfies conditions (1) and (2). It remains to show that $\mathbf{h}^*$ also satisfies the transversality condition (3). Replace c_t^* by $h_t^{*(\alpha+\gamma)}\psi\left(\dfrac{h_{t+1}^*}{h_t^*}\right)$ and $u'(c)$ by $\mu c^{\mu-1}$. Since

$$\beta^t h_t^{*(\alpha+\gamma)\mu}\psi\left(\frac{h_{t+1}^*}{h_t^*}\right)^{\mu-1}\frac{h_{t+1}^*}{h_t^*}\psi'\left(\frac{h_{t+1}^*}{h_t^*}\right)\leqslant h_0^{(\alpha+\gamma)\mu}\psi(\nu^*)^{\mu-1}\nu^*\psi'(\nu^*)\left[\beta(1+\lambda)^{(\alpha+\gamma)\mu}\right]^t,$$

and $\beta(1+\lambda)^{(1+\gamma)\mu}<1$, we have

$$\lim_{t\to\infty}\beta^t h_t^{*(\alpha+\gamma)\mu}\psi\left(\frac{h_{t+1}^*}{h_t^*}\right)^{\mu-1}\frac{h_{t+1}^*}{h_t^*}\psi'\left(\frac{h_{t+1}^*}{h_t^*}\right)=0,$$

which is the condition (3).

The proof of the uniqueness is rather long. It will be done in three steps. The idea is to prove that, for any solution to EULER equation different from the one which grows at rate ν^*, the rate of growth will converge to $1+\lambda$. This property is crucial to prove that this solution does not satisfy the transversality condition and, from the previous proposition, is not optimal. One obviously concludes that the equilibrium is unique and grows at rate ν^*.

Step 1. Let $\nu_t=\dfrac{h_{t+1}}{h_t}$ and $\delta=1/(1-\mu)$. Euler equation can be written as:

$$\frac{\psi(\nu_{t+1})}{(\alpha\psi(\nu_{t+1})-\nu_{t+1}\psi'(\nu_{t+1}))^\delta}=\beta^\delta\frac{\psi(\nu_t)}{(-\nu_t^{1-(\alpha+\gamma)\mu}\psi'(\nu_t))^\delta},\tag{2}$$

or

$$\Phi(\nu_{t+1})=\psi(\nu_t),\tag{3}$$

with $\Phi(x)=\dfrac{\psi(x)}{(\alpha\psi(x)-x\psi'(x))^\delta}$ and $\psi(x)=\beta^\delta\dfrac{\psi(x)}{(-x^{1-(\alpha+\gamma)\mu}\psi'(x))^\delta}$.

We will show that $\nu_{t+1} = I(\nu_t)$ with $I' > 0$. Indeed, tedious computations give

$$\Phi'(x) = \frac{\psi'(x)(\alpha\psi(x) - x\psi'(x)) + \delta\psi(x)((1-\alpha)\psi'(x) + x\psi''(x)}{(\psi(x) - x\psi'(x))^{1+\delta}} < 0,$$

and

$$\psi'(x) = \beta^{\delta} \frac{-\psi'(x)^2 x^{1-(\alpha+\gamma)\mu} + \delta\psi(x)\left[(1-(\alpha+\gamma)\mu)x^{-(\alpha+\gamma)\mu}\psi'(x) + x^{1-(\alpha+\gamma)\mu}\psi''(x)\right]}{(-x^{1-(1+\gamma)\mu}\psi'(x))^{1+\delta}} < 0.$$

Hence, one can write $\nu_{t+1} = I(\nu_t)$ with $I' > 0$.

Observe that EULER equation (2) has only two fixed points which are ν^* and $1+\lambda$. We have shown that the sequence $\mathbf{h}^*$ with $h_t^* = (\nu^*)^t h_0, \forall t$, is an equilibrium. The sequence $\mathbf{h}$ with $h_t = (1+\lambda)^t h_0, \forall t$, is not optimal since the associated consumptions equal zero at every date.

Step 2. Consider a non stationary sequence ν which satisfies EULER equation (2) and $\forall t, 1 \leqslant \nu_t \leqslant 1+\lambda$. We will show that such a sequence converges to $1+\lambda$. In view of the monotonicity of I, since $\nu_1 \leqslant \nu_0$ implies that $\nu_2 = I(\nu_1) \leqslant I(\nu_0) = \nu_1$ (respectively $\nu_1 \geqslant \nu_0$ implies that $\nu_2 \geqslant \nu_1$), by an easy induction, the sequence ν is weakly monotone. Hence it is converging to a fixed-point of I: either ν^* or $1+\lambda$.

We will show that to assume that ν converges to ν^* leads to a contradiction. Indeed, let $\varepsilon_t = \nu^* - \nu_t$. First, observe that $\nu_0 \neq \nu^*$ implies that for all t, $\varepsilon_t \neq 0$. When $t \to +\infty$, $\varepsilon_{t+1} \sim I'(\nu^*)\varepsilon_t$. Tedious computations show that $I'(\nu^*) > 1$.

In particular, for t large enough, the sequence $(|\varepsilon_t|)$ is increasing, which contradicts $\nu_t \to \nu^*$.

Step 3. When $u(c) = c^{\mu}$ and $c_t = h_t^{\gamma}\left(h_t^{\alpha}\psi\left(\frac{h_{t+1}}{h_t}\right)\right)$, the transversality condition becomes

$$\lim_{t\to\infty} \beta^t h_t^{(\alpha+\gamma)\mu}\psi\left(\frac{h_{t+1}}{h_t}\right)^{\mu-1}\psi'\left(\frac{h_{t+1}}{h_t}\right)\frac{h_{t+1}}{h_t} = 0.$$

Since $\frac{h_{t+1}}{h_t} \to 1+\lambda$ when $t \to \infty$, and $\psi'(1+\lambda) > -\infty$, the transversality condition is equivalent to $\lim_{t\to\infty} \beta^t h_t^{(\alpha+\gamma)\mu}\psi\left(\frac{h_{t+1}}{h_t}\right)^{\mu-1} = 0$.

Let us denote by $\nu_t = \frac{h_{t+1}}{h_t}$, $\varepsilon_t = 1+\lambda-\nu_t$ and

$S_t = \beta^t h_t^{(\alpha+\gamma)\mu}\psi(\nu_t)^{\mu-1}$. When $t \to \infty$, then $\nu_t \to 1+\lambda$, and $\varepsilon_t \to 0$. Consequently

$$\psi(\nu_t) = \psi(\nu_t) - \psi(1+\lambda) \sim -\psi'(1+\lambda)(1+\lambda-\nu_t) = -\psi'(1+\lambda)\varepsilon_t.$$

It follows that $S_t \sim \widehat{S}_t(-\psi'(1+\lambda))^{\mu-1}$ where $\widehat{S}_t = \beta^t h_t^{(\alpha+\gamma)\mu}(\varepsilon_t)^{\mu-1}$. Hence, in order to prove that the transversality does not hold, we will prove that $\lim_{t\to\infty} \widehat{S}_t > 0$. We have

$$\begin{aligned}\varepsilon_{t+1} &= (1+\lambda) - \nu_{t+1} = I(1+\lambda) - I(\nu_t) \sim I'(1+\lambda)(1+\lambda-\nu_t)\\ &= I'(1+\lambda)\varepsilon_t.\end{aligned}$$

Let us now remark that $I'(1+\lambda) < 1$ and this imply in particular the summability of (ε_t). Indeed, we obtain, after tedious computations:

$$I'(1+\lambda) = \left[\beta(1+\lambda)^{(\alpha+\gamma)\mu}\right]^{\frac{1}{1-\mu}} < 1.$$

Letting $\pi_t = \widehat{S}_{t+1}/\widehat{S}_t$, with classical notations, we can write

$$\begin{aligned}\pi_t &= \beta\nu_t^{(\alpha+\gamma)\mu}(\varepsilon_{t+1}/\varepsilon_t)^{\mu-1}\\ &= \beta(1+\lambda-\varepsilon_t)^{(\alpha+\gamma)\mu}(I'(1+\lambda) + (1/2)I''(1+\lambda)\varepsilon_t + o(\varepsilon_t))^{\mu-1}\end{aligned}$$

In view of the computation of $I'(1+\lambda)$,

$$\pi_t = \left(\frac{1+\lambda-\varepsilon_t}{1+\lambda}\right)^{(\alpha+\gamma)\mu}\left(1 + \frac{I''(1+\lambda)\varepsilon_t}{2I'(1+\lambda)} + o(\varepsilon_t)\right)^{\mu-1}$$

where $o(\varepsilon_t)/\varepsilon_t \to 0$, when $t \to \infty$.

Therefore, the sequence $(\ln(\pi_t)/\varepsilon_t)$ converges. The summability of (ε_t) implies the summability of $(\ln\pi_t)$ which is equivalent to the convergence of the infinite product $(\pi_0 \cdots \pi_t)$ to a positive limit. Since $\widehat{S}_{t+1} = (\pi_0\pi_1...\pi_t)\widehat{S}_0$, we proved that $\widehat{S}$ does not tend to 0.

4 Concluding Remarks

1. In our paper, we prove that health care helps the economy to take off if the quality and/or the external effect of the human capital are high enough. Assumption $H7$ ensures this condition.

2. We also prove that there exists a unique optimal path and it grows with a constant rate. This one positively depends on the quality and/or the external effect of the human capital.

3. In LUCAS [1988] we have a competitive equilibrium with constant growth rate. Here we obtain more. This competitive equilibrium is the unique one. It exists under conditions (*e.g.* $H7b$, $H8b$) which are more stringent than those (*e.g.* $H7$) required for the existence of increasing optimal paths. It is also interesting to notice that Assumption $H8b$ is in the spirit of the result in LE VAN *et alii* [2002], Proposition 3: a competitive equilibrium exists if the externality effect of the human capital is not large, or, in other words, if the production function of the social planner does not exhibit too much increasing returns. ▼

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