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A Simple Version of the Lucas Model

Mazamba Tédie

Abstract

This discrete-time version of the Lucas model dot not include the physical capital. We intregrate in the utility function the leisure time. We examine the social planer and the competitive equilibrium. The main conclusions are that the consumer always chooses to train, the human capital growth rate increases with the externality and the quality of training, and that the equilibrium defined by Lucas (1988) is a competitive equilibrium under some conditions.

1. Introduction

This model is a discrete-time version of the model of Lucas without physical capital. The consumer devotes the fraction θ of his non-leisure time to current production and the remaining $(1-\theta)$ to human capital accumulation. We consider that the utility of consumer increases with his leisure time. This assumption implies that the utility increases with the human capital accumulation that is with the training. Following Lucas (1988), the human capital has : 1- an *external effect* through the externality. 2- an *internal effect* which increases the productivity through the medium of training.

This paper is organized into seven sections. Section 2 introduces assumptions and examines the social planer problem. After defining the *equilibrium* (according to Lucas and Romer) and *competitive equilibrium*, section 3 shows that an equilibrium is a competitive equilibrium. The following sections conclude and give some proofs.

2. Social Planer

The utility function is concave ($0 < \mu < 1$ and $0 < \zeta < 1$) :

$$\max \sum_{t=0}^{+\infty} \beta^t c_t^{\mu} (1-\theta_t)^{\zeta}$$

Subject to,

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$$\forall t \geq 0,\ 0 \leq c_t \leq h_t^{\gamma}(\theta_t h_t)^{\alpha}$$
$$h_{t+1} = h_t(1 + \lambda\phi(1 - \theta_t))$$
$$0 < \alpha < 1,\ \gamma \geq 0,\ 0 \leq \theta_t \leq 1,\ h_0 > 0 \text{ given}$$

We make the following assumptions :

H1 : ϕ is concave, increasing and twice continuously differentiable. $\phi(0) = 0, \phi(1) = 1$, $\lambda > 0$ and $\phi'(0) > 1$.

H2 : $0 < \beta(1+\lambda)^{(\alpha+\gamma)\mu} < 1$.

The parameter λ balanced the technology of training. Let us define the function $\psi : [1, 1+\lambda] \rightarrow [0, 1]$ by $\psi(x) = 1 - \phi^{-1}\left(\frac{1}{\lambda}(x-1)\right)$. Where ϕ^{-1} denotes the inverse function of ϕ. ψ is clearly decreasing. It is easy to verify that : $\psi(1) = 1$ and $\psi(1+\lambda) = 0$. This function gives the working time when the human capital grows by factor x. ψ is continuously differentiable, decreasing, with $\psi'(1) = -\frac{1}{\lambda\phi'(0)}$, $\psi'(1+\lambda) = -\frac{1}{\lambda\phi'(1)}$ and concave.

The problem becomes :

$$\max \sum_{t=0}^{+\infty} \beta^t h_t^{(\alpha+\gamma)\mu} \left(\psi(\frac{h_{t+1}}{h_t})\right)^{\alpha\mu} \left(1 - \psi(\frac{h_{t+1}}{h_t})\right)^{\zeta}$$

Subject to :

$$\forall t \geq 0,\ h_t \leq h_{t+1} \leq h_t(1+\lambda) \text{ and } h_0 > 0 \text{ given.}$$

Proposition 1 *Under **H1-H2**, there exists a solution.*

Proof. See the appendix 1.

Proposition 2 *Each optimal path of human capital* $\mathbf{h} = (h_0, h_1, ..., h_t, ...)$ *verifies* $h_0 < h_1 < \cdots < h_t < \cdots$.

Proof. See the appendix 2.

Proposition 3 *Under assumptions **H1,H2** and **H3** :* $(\alpha+\gamma)\mu - 1 < 0$ *:*

(a) *The optimal path of human capital has a constant growth rate, strictly positive and which increases with parameter* γ.

(b) *The optimal path of human capital is an increasing function of* λ.

Proof. We prove (a) in several stages.

1. Let V be the *Value Function* of our problem of optimal growth :

$$V(h_0) = \max \sum_{t=0}^{+\infty} \beta^t h_t^{(\alpha+\gamma)\mu} \left(\psi(\frac{h_{t+1}}{h_t})\right)^{\alpha\mu} \left(1 - \psi(\frac{h_{t+1}}{h_t})\right)^{\zeta}$$

Under the constraints : $\forall t \geq 0, h_t \leq h_{t+1} \leq h_t(1+\lambda),$ and $h_0 > 0$ given.
This value function satisfied (see Le Van & Morhaim 2002) :

$$V(h_0) = Ah_0^{(\alpha+\gamma)\mu}$$

Let us consider h_0, the optimal value h_1 of the human capital at date 1 is the solution of the following equation :

$$h_0^{(\alpha+\gamma)\mu} \max_{y\in[h_0,(1+\lambda)h_0]} \left\{ \left(\psi(\tfrac{y}{h_0})\right)^{\alpha\mu} \left(1-\psi(\tfrac{y}{h_0})\right)^{\zeta} + \beta A \left(\tfrac{y}{h_0}\right)^{(\alpha+\gamma)\mu} \right\}$$

We can see that $h_1^* = \nu h_0$ where ν is the solution of the equation : $\max\left\{(\psi(z))^{\alpha\mu}(1-\psi(z))^{\zeta} + \beta A(\gamma) z^{(\alpha+\gamma)\mu}\right\}$. Since the problem is stationary, if $\{h_t\}$ is the optimal path, then : $h_t = \nu^t h_0$, $\forall t$.

2. We know that the human capital path verifies $h_{t+1} > h_t$, $\forall t \geq 0$. The Euler equation is given by :

$$\begin{aligned}
&h_t^{(\alpha+\gamma)\mu-1}\psi'(\tfrac{h_{t+1}}{h_t})\Psi(\tfrac{h_{t+1}}{h_t}) \left[\alpha\mu\left(1-\psi(\tfrac{h_{t+1}}{h_t})\right) - \zeta\psi(\tfrac{h_{t+1}}{h_t})\right] \\
&= -\beta(\alpha+\gamma)\mu h_{t+1}^{(\alpha+\gamma)\mu-1}\Psi(\tfrac{h_{t+2}}{h_{t+1}}) \left(\psi(\tfrac{h_{t+2}}{h_{t+1}})\right)\left(1-\psi(\tfrac{h_{t+2}}{h_{t+1}})\right) \\
&+\beta h_{t+1}^{(\alpha+\gamma)\mu-1}\tfrac{h_{t+2}}{h_{t+1}}\psi'(\tfrac{h_{t+2}}{h_{t+1}})\Psi(\tfrac{h_{t+2}}{h_{t+1}}) \left[\alpha\mu\left(1-\psi(\tfrac{h_{t+2}}{h_{t+1}})\right) - \zeta\psi(\tfrac{h_{t+2}}{h_{t+1}})\right]
\end{aligned}$$

With $\Psi(\frac{h_{t+1}}{h_t}) = \left(\psi(\frac{h_{t+1}}{h_t})\right)^{\alpha\mu-1}\left(1-\psi(\frac{h_{t+1}}{h_t})\right)^{\zeta-1}$.
This equation gives the human capital growth rate that is constant (ν) :

$$1 = \frac{\zeta}{\alpha\mu}\frac{\psi(\nu)}{1-\psi(\nu)}\left(1-\beta\nu^{(\alpha+\gamma)\mu}\right) - \frac{\beta}{\alpha}(\alpha+\gamma)\nu^{(\alpha+\gamma)\mu-1}\frac{\psi(\nu)}{\psi'(\nu)} + \beta\nu^{(\alpha+\gamma)\mu}$$

Let $F(\nu) = \frac{\zeta}{\alpha\mu}\frac{\psi(\nu)}{1-\psi(\nu)}\left(1-\beta\nu^{(\alpha+\gamma)\mu}\right)$ with $G(\nu) = -\frac{\beta}{\alpha}(\alpha+\gamma)\nu^{(\alpha+\gamma)\mu}\frac{\psi(\nu)}{\psi'(\nu)} + \beta\nu^{(\alpha+\gamma)\mu}$. Functions F and G are decreasing since : $F'(x) = \left(1-\beta x^{(\alpha+\gamma)\mu}\right)\frac{\zeta}{\alpha\mu}\frac{\psi'(x)}{(1-\psi(x))^2} - \frac{\beta(\alpha+\gamma)}{\alpha}\frac{\psi(x)}{1-\psi(x)}x^{(\alpha+\gamma)\mu-1} < 0$, $G'(x) = -\frac{\beta}{\alpha}(\alpha+\gamma)x^{(\alpha+\gamma)\mu-2}\left[\frac{\psi(x)}{\psi'(x)}\left(((\alpha+\gamma)\mu-1) - \frac{\psi''(x)}{\psi(x)}x\right) + x\left(\frac{1}{\alpha}-\mu\right)\right] < 0$. Moreover, $F(1+\lambda) = 0$, $\lim_{x\to 1} F(x) = +\infty$, $G(1) = \frac{\beta(\alpha+\gamma)}{\alpha}\lambda\phi'(0)$ and $G(1+\lambda) = \beta(1+\lambda)^{(\alpha+\gamma)\mu} < 1$ according to **H2**. Hence, there exists a unique solution $\nu \in]1, 1+\lambda[$.

3. We know that the value function verifies the Bellman equation :

$$V(h) = h^{(\alpha+\gamma)\mu}\max_{\nu\in[1,1+\lambda]}\left\{(\psi(\nu))^{\alpha\mu}(1-\psi(\nu))^{\zeta} + \beta A(\gamma)\nu^{(\alpha+\gamma)\mu}\right\}$$

The derivate of function $(\psi(\nu))^{\alpha\mu}(1-\psi(\nu))^{\zeta} + \beta A(\gamma)\nu^{(\alpha+\gamma)\mu}$ is cancelled :

$$\begin{aligned}
&-\alpha\mu\,\psi'(\nu^*)(\psi(\nu^*))^{\zeta} + \zeta\,(\nu^*)(\psi(\nu^*))^{\alpha\mu}(1-\psi(\nu^*))^{\zeta-1} \\
&= \beta A(\gamma)(\alpha+\gamma)\mu\nu^{*(\alpha+\gamma)\mu-1}
\end{aligned}$$

When γ increases, the graph of the function $\beta A(\gamma)(\alpha+\gamma)\mu\nu^{*(\alpha+\gamma)\mu-1}$ moves to the top while the left-hand side remains constant. Consequently, the growth rate increases with the parameter of the externality. This ends the proof of the claim (a).

4. Let us rewrite the Euler equation : $1 = F_\lambda(x) + G_\lambda(x)$. Note that $\lambda < \lambda' \Rightarrow \psi_\lambda < \psi_{\lambda'}$ and $-\psi'_\lambda < -\psi'_{\lambda'}$. Hence, F and G are increasing with λ. Moreover, F and G are decreasing with ν, then :

$$\frac{d\nu}{d\lambda} = -\left[\left(\frac{\partial F}{\partial\lambda} + \frac{\partial G}{\partial\lambda}\right) / \left(\frac{\partial G}{\partial\nu} + \frac{\partial F}{\partial\nu}\right)\right] > 0$$

3. Equilibrium and Competitive Equilibrium

We introduce the concepts of *equilibrium* (according to Lucas and Romer) and *competitive equilibrium*. Take a human capital path $\bar{\mathbf{h}} = (\bar{h}_1, ..., \bar{h}_t, ...)$ to be given. Given $\bar{\mathbf{h}}$, consider the problem :

$$\max_{c_t} \sum_{t=0}^{+\infty} \beta^t u(c_t, \theta_t)$$

Under the constraints,

$$\begin{gathered} \forall t,\ 0 \le c_t \le G(\bar{h}) f(\theta_t h_t) \\ h_{t+1} = h_t(1 + \lambda\phi(1 - \theta_t)) \\ 0 \le \theta_t \le 1,\ h_0 > 0 \text{ given} \end{gathered}$$

The solution $\mathbf{h} = (h_0, h_1, ..., h_t, ...)$ of this model depends on $\bar{\mathbf{h}}$. In others words, $\mathbf{h} = \Phi(\bar{\mathbf{h}})$. A *equilibrium* is a human capital path $\mathbf{h}^* = (h_0, ..., h_t^*, ...)$ such that $\mathbf{h}^* = \Phi(\mathbf{h}^*)$. In order to define a *competitive equilibrium*, we need before to define the space of the prices which supports this equilibrium. Observe that all feasible paths of consumption $\mathbf{c}$ verify for all $t : 0 \le c_t \le h_t^{\alpha+\gamma}$ with $h_t \le h_0(1+\lambda)^t$. In others words, $\mathbf{c}$ belongs to :

$$\ell^\infty = \left\{ \mathbf{c} : \sup_{t=0,..,+\infty} \frac{|c_t|}{(1+\lambda)^{(\alpha+\gamma)t}} < +\infty \right\}$$

Let ℓ_+^∞ be the set of non negative sequences of ℓ^∞. The price sequence p_t is such as all consumption paths c_t verify $\sum_{t=0}^{+\infty} p_t c_t < +\infty$. Likewise, the wage path w_t is such as $\sum_{t=0}^{\infty} w_t h_t < +\infty$. In order to satisfy these two conditions, we must take the prices space and the wages space as follows :

$$\ell_p^1 = \left\{ \mathbf{p} : \sum_{t=0}^{+\infty} |p_t| (1+\lambda)^{(\alpha+\gamma)t} < +\infty \right\} ;\ \ell_w^1 = \left\{ \mathbf{w} : \sum_{t=0}^{+\infty} |w_t| (1+\lambda)^t < +\infty \right\}$$

Let us denote ℓ_+^1, the set of non-negative sequences of ℓ^1.

We define a *competitive equilibrium* for the model of Lucas.
A collection of sequences $(\mathbf{h}^*, \mathbf{c}^*, \boldsymbol{\theta}^*, \mathbf{p}^*, \mathbf{w}^*)$ is a competitive equilibrium if :

1. $(\mathbf{c}^*, \boldsymbol{\theta}^*)$ is a solution of the consumer program :

$$\max_{c_t, \theta_t} \sum_{t=0}^{+\infty} \beta^t u(c_c, \theta_t)$$

   Under the constraints,

$$\begin{gathered} \sum_{t=0}^{+\infty} p_t^* c_t \le \sum_{t=0}^{+\infty} w_t^* \theta_t h_t + \Pi^* \\ \forall t \ge 0,\ \theta_t = \psi\left(\tfrac{h_{t+1}}{h_t}\right),\ h_0 > 0 \text{ given} \end{gathered}$$

2. $\boldsymbol{\theta}^*$ is a solution of the firm program :

$$\Pi^* = \max_{\theta} \left\{ \sum_{t=0}^{+\infty} p_t^* (h_t^*)^\gamma (\theta_t h_t^*)^\alpha - \sum_{t=0}^{+\infty} w_t^* \theta_t h_t^* \right\}$$

3. Equilibrium on the goods and services market :

$$\forall t \geq 0,\ c_t^* = (h_t^*)^\gamma (\theta_t^* h_t^*)^\alpha$$

Proposition 4 $\mathbf{h}^*$ *is a equilibrium from* $h_0 > 0$ *if and only if it verifies the three following conditions :*

1. *Interiority :* $\forall t \geq 0,\ h_t^* < h_{t+1}^* < (1+\lambda) h_t^*,\ \ h_0^* = h_0 > 0$

2. *Euler equation (*$\forall t \geq 0$*),*

$$\begin{aligned}
&\alpha\mu\, h_t^{*(\alpha+\gamma)\mu-1} \psi'(\tfrac{h_{t+1}^*}{h_t^*}) \left(\psi(\tfrac{h_{t+1}^*}{h_t^*})\right)^{\alpha\mu-1} \left(1-\psi(\tfrac{h_{t+1}^*}{h_t^*})\right)^{\zeta} \\
&-\zeta\, h_t^{*(\alpha+\gamma)\mu-1} \psi'(\tfrac{h_{t+1}^*}{h_t^*}) \left(\psi(\tfrac{h_{t+1}^*}{h_t^*})\right)^{\alpha\mu} \left(1-\psi(\tfrac{h_{t+1}^*}{h_t^*})\right)^{\zeta-1} \\
&+\beta\alpha\mu\, h_{t+1}^{*(\alpha+\gamma)\mu-1} \left(\psi(\tfrac{h_{t+2}^*}{h_{t+1}^*})\right)^{\alpha\mu} \left(1-\psi(\tfrac{h_{t+2}^*}{h_{t+1}^*})\right)^{\zeta} \\
&-\beta\alpha\mu\, h_{t+1}^{*(\alpha+\gamma)\mu-1} \tfrac{h_{t+2}^*}{h_{t+1}^*} \psi'(\tfrac{h_{t+2}^*}{h_{t+1}^*}) \left(\psi(\tfrac{h_{t+2}^*}{h_{t+1}^*})\right)^{\alpha\mu-1} \left(1-\psi(\tfrac{h_{t+2}^*}{h_{t+1}^*})\right)^{\zeta} \\
&+\beta\zeta\, h_{t+1}^{*(\alpha+\gamma)\mu-1} \tfrac{h_{t+2}^*}{h_{t+1}^*} \psi'(\tfrac{h_{t+2}^*}{h_{t+1}^*}) \left(\psi(\tfrac{h_{t+2}^*}{h_{t+1}^*})\right)^{\alpha\mu} \left(1-\psi(\tfrac{h_{t+2}^*}{h_{t+1}^*})\right)^{\zeta-1} = 0
\end{aligned}$$

3. *Transversality condition,*

$$\begin{aligned}
\lim_{t\to+\infty} \beta^t h_t^{*(\alpha+\gamma)\mu-1} \psi'(\frac{h_{t+1}^*}{h_t^*}) \left(\psi(\frac{h_{t+1}^*}{h_t^*})\right)^{\alpha\mu-1} \left(1-\psi(\frac{h_{t+1}^*}{h_t^*})\right)^{\zeta-1} \\
\left[\alpha\mu\left(1-\psi(\tfrac{h_{t+1}^*}{h_t^*})\right) - \zeta\psi(\tfrac{h_{t+1}^*}{h_t^*})\right] h_{t+1}^* = 0
\end{aligned}$$

Proof. See the appendix 3.

Proposition 5 *Under the assumptions of proposition 3 and* ***H4*** *:* $\lambda \leq \frac{1}{\beta} - 1$*, there exists an equilibrium* $\mathbf{h}^*$ *which increases at constant rate* ν*. The equilibrium growth rate* $\mathbf{h}^*$ *is weaker than that of the centralized rate. We can associate with this equilibrium the stationary sequence* $\boldsymbol{\theta^*} = (\psi(\nu))$*, a consumption sequence* $\mathbf{c}^*$*, a price system* $\mathbf{p}^*$*, wage* $\mathbf{w}^*$ *such as the collection of sequences* $(\mathbf{h}^*, \mathbf{c}^*, \boldsymbol{\theta^*}, \mathbf{p}^*, \mathbf{w}^*)$ *is a competitive equilibrium.*

Proof. 1. We know that if $\mathbf{h}^*$ is an equilibrium then it verifies interiority, the Euler equation and the transversality condition. In addition, let us show that exists a human capital sequence that increases at constant rate and satisfies the Euler equation. Indeed, according to Euler equation, this rate ν must satisfy :

$$1 = \frac{\zeta}{\alpha\mu} \frac{\psi(\nu)}{1-\psi(\nu)} \left(1 - \beta\nu^{(\alpha+\gamma)\mu}\right) - \beta\nu^{(\alpha+\gamma)\mu-1} \frac{\psi(\nu)}{\psi'(\nu)} + \beta\nu^{(\alpha+\gamma)\mu} \equiv V(\nu)$$

Let $F(\nu) = \frac{\zeta}{\alpha\mu}\frac{\psi(\nu)}{1-\psi(\nu)}\left(1-\beta\nu^{(\alpha+\gamma)\mu}\right)$ and $H(\nu) = -\beta\nu^{(\alpha+\gamma)\mu-1}\frac{\psi(\nu)}{\psi'(\nu)} + \beta\nu^{(\alpha+\gamma)\mu}$. We know that F is decreasing, $\lim_{x\to 1} F(x) = +\infty$ and that $F(1+\lambda) = 0$. We show that H is also decreasing :
$H'(\nu) = -\beta\nu^{(\alpha+\gamma)\mu-2}\left[((\alpha+\gamma)\mu-1)\left(\frac{\psi(\nu)}{\psi'(\nu)}-\nu\right) - \nu\frac{\psi(\nu)\psi''(\nu)}{(\psi'(\nu))^2}\right] < 0$. One has $V(x) = F(x)+H(x)$, $V'(x) = F'(x)+H'(x)$, $\lim_{x\to 1} V(x) = \lim_{x\to 1} F(x) + \lim_{x\to 1} H(x) = +\infty$ and $V(1+\lambda) = F(1+\lambda)+H(1+\lambda) = \beta(1+\lambda)^{(\alpha+\gamma)\mu} < 1$ according to **H2**. Consequently, there exists a unique solution ν which belongs to $]1, 1+\lambda[$. It's easy to show that this rate is weaker than the rate of social planer program which is the solution of the equation : $1 = F(\nu) + G(\nu)$, since $G(\nu) = H(\nu) - \frac{\beta\gamma}{\alpha}\nu^{(\alpha+\gamma)\mu-1}\frac{\psi(\nu)}{\psi'(\nu)}$. Let h^* be the trajectory defined by : $h_0^* = h_0$, $h_{t+1}^* = \nu h_t^*$, $\forall t$. Obviously, it satisfies the interiority and Euler equation. We must show than it verifies the transversality condition to conclude that h^* is an equilibirum. Now,

$$\begin{aligned}&\beta^t h_t^{*(\alpha+\gamma)\mu}\frac{h_{t+1}^*}{h_t^*}\psi'(\frac{h_{t+1}^*}{h_t^*})\left(\psi(\frac{h_{t+1}^*}{h_t^*})\right)^{\alpha\mu-1}\left(1-\psi(\frac{h_{t+1}^*}{h_t^*})\right)^{\zeta-1}A(\frac{h_{t+1}^*}{h_t^*})\\ &= \beta^t h_0^{(\alpha+\gamma)\mu}\nu^{(\alpha+\gamma)\mu t}\psi'(\nu)\nu(\psi(\nu))^{\alpha\mu-1}(1-\psi(\nu))^{\zeta-1}A(\nu)\\ &\le h_0^{(\alpha+\gamma)\mu}(\psi(\nu))^{\alpha\mu-1}\nu\psi'(\nu)(1-\psi(\nu))^{\zeta-1}A'\left[\beta(1+\lambda)^{(\alpha+\gamma)\mu}\right]^t\end{aligned}$$

Where $A' = \alpha\mu\left(1-\psi(\frac{h_{t+1}^*}{h_t^*})\right) - \zeta\psi(\frac{h_{t+1}^*}{h_t^*})$. Assumption **H2** implies :

$$\lim_{t\to+\infty}\beta^t h_t^{*(\alpha+\gamma)\mu-1}\psi'(\frac{h_{t+1}^*}{h_t^*})\left(\psi(\frac{h_{t+1}^*}{h_t^*})\right)^{\alpha\mu-1}\left(1-\psi(\frac{h_{t+1}^*}{h_t^*})\right)^{\zeta-1}A' = 0$$

This is the transversality condition.

2. We show that this trajectory is a competitive equilibrium. Let us define the price path and the wage path, $\mathbf{p}^*$, $\mathbf{w}^*$ by :

$$\begin{aligned}p_t^* &= \beta^t\frac{\partial u(c_t,\theta_t)}{\partial c_t} = \mu\beta^t h_t^{*(\alpha+\gamma)(\mu-1)}(\psi(\nu))^{\alpha(\mu-1)}(1-\psi(\nu))^{\zeta}\\ w_t^* &= \beta^t h_t^{*(\alpha+\gamma)\mu-1}(\psi(\nu))^{\alpha\mu-1}(1-\psi(\nu))^{\zeta-1}[\alpha\mu(1-\psi(\nu))-\zeta\psi(\nu)]\end{aligned}$$

Where $h_t^* = \nu^t h_0$.
a) It is easy to see that the sequence $\boldsymbol{\theta}^*$ defined by $\theta_t^* = \psi(\nu)$, for all t, maximizes the profit of the enterprise according to $\mathbf{p}^*$ and $\mathbf{w}^*$.
b) In order to prove that the consumption path and the working time path (c_t^*, θ_t^*) maximize the consumer utility, consider :

$$\Delta_T = \sum_{t=0}^{T}\beta^t u(c_t^*,\theta_t^*) - \sum_{t=0}^{T}\beta^t u(c_t,\theta_t)$$

Since $\sum_{t=0}^{+\infty}\beta^t u'(c_t)c_t^* = \sum_{t=0}^{+\infty} w_t^*\theta_t^* h_t^* + \Pi^*$ and $\sum_{t=0}^{+\infty}\beta^t u'(c_t^*)c_t < \sum_{t=0}^{+\infty} w_t^*\theta_t h_t + \Pi^*$ with $\theta_t = \psi(h_{t+1}/h_t)$, one has :

$$\begin{aligned}\Delta_T \ge \sum_{t=0}^{T}\beta^t&\left[(h_t^*-h_t)\left(\alpha\mu h_t^{*(\alpha+\gamma)\mu-1}\left(1-\psi(\frac{h_{t+1}^*}{h_t^*})\right)\Phi(\frac{h_{t+1}^*}{h_t^*})\right)\right.\\ &+\zeta h_t^{*(\alpha+\gamma)\mu-1}\frac{h_{t+1}^*}{h_t^*}\left(\psi(\frac{h_{t+1}^*}{h_t^*})\right)^{\alpha\mu}\left(1-\psi(\frac{h_{t+1}^*}{h_t^*})\right)^{\zeta-1}\psi'(\frac{h_{t+1}^*}{h_t^*})\Bigg)\\ &+(h_{t+1}^*-h_{t+1})\left(\alpha\mu h_t^{*(\alpha+\gamma)\mu-1}\left(\psi(\frac{h_{t+1}^*}{h_t^*})\right)^{\alpha\mu-1}\left(1-\psi(\frac{h_{t+1}^*}{h_t^*})\right)^{\zeta}\psi'(\frac{h_{t+1}^*}{h_t^*})\right.\\ &\left.\left.-\zeta h_t^{*(\alpha+\gamma)\mu-1}\left(\psi(\frac{h_{t+1}^*}{h_t^*})\right)^{\alpha\mu}\left(1-\psi(\frac{h_{t+1}^*}{h_t^*})\right)^{\zeta-1}\psi'(\frac{h_{t+1}^*}{h_t^*})\right)\right]\end{aligned}$$

Where $\Phi(\frac{h^*_{t+1}}{h^*_t}) = \left(\psi(\frac{h^*_{t+1}}{h^*_t})\right)^{\alpha\mu-1}\left(1-\psi(\frac{h^*_{t+1}}{h^*_t})\right)^{\zeta-1}\left(\psi(\frac{h^*_{t+1}}{h^*_t}) - \frac{h^*_{t+1}}{h^*_t}\psi'(\frac{h^*_{t+1}}{h^*_t})\right)$.
Using the Euler equation, we obtain :

$$\Delta_T \geq \beta^T h_T^{*(\alpha+\gamma)\mu-1}\left(\psi(\frac{h^*_{T+1}}{h^*_T})\right)^{\alpha\mu-1}\left(1-\psi(\frac{h^*_{T+1}}{h^*_T})\right)^{\zeta-1}$$
$$\left[\alpha\mu\left(1-\psi(\frac{h^*_{T+1}}{h^*_T})\right) - \zeta\psi(\frac{h^*_{T+1}}{h^*_T})\right]\psi'(\frac{h^*_{T+1}}{h^*_T}) = \beta^T w^*_T\psi'(\frac{h^*_{T+1}}{h^*_T})h^*_{T+1}$$

By definition of w^*_T. According to the transversality condition, we conclude that $\lim_{T\to+\infty}\Delta_T \geq 0$.
c) The goods market is balanced since for all t : $c^*_t = (h^*_t)^\gamma(\theta^*_t h^*_t)^\alpha$.
d) To complete this proof, let us show that $\mathbf{p}^*$ belongs to ℓ^1_p and $\mathbf{w}^*$ belongs to ℓ^1_w. One has :

$$\sum_{t=0}^{+\infty} p_t(1+\lambda)^{(\alpha+\gamma)t} < \mu B' h_0^{(\alpha+\gamma)(\mu-1)}\sum_{t=0}^{+\infty}\left[\beta(1+\lambda)^{(\alpha+\gamma)\mu}\right]^t < +\infty$$

According to **H2** and with $B' = (\psi(\nu))^{\alpha(\mu-1)}(1-\psi(\nu))^\zeta$. Likewise,

$$\sum_{t=0}^{+\infty} w_t(1+\lambda)^t = C' h_0^{(\alpha+\gamma)\mu-1}\sum_{t=0}^{+\infty}\left[\beta(1+\lambda)\nu^{(\alpha+\gamma)\mu-1}\right]^t < +\infty$$

According to **H4**, $1 < \nu < 1+\lambda$ and where $C' = (\psi(\nu))^{\alpha\mu-1}(1-\psi(\nu))^{\zeta-1}[\alpha\mu(1-\psi(\nu)) - \zeta\psi(\nu)]$. This ends the proof. A collection of paths $(\mathbf{h}^*, \mathbf{c}^*, \boldsymbol{\theta}^*, \mathbf{p}^*, \mathbf{w}^*)$ is a competitive equilibrium.

4. Conclusion

This dicrete-time version of the Lucas model solves the social planer program and shows that an equilibrium for this model is a competitive equilibrium. Moreover, the model concludes that : 1- when the utility depends on consumption and leisure time, the consumer always prefers to increase his skill level. 2- the quality of training increases the human capital growth rate. 3- the externality is related positively to the human capital growth rate through it contribution to the productivity of all factors of production.

5. Appendix 1

It's easy to verify that if $\mathbf{c} = (c_0, c_1, ..., c_t, ...)$ is a feasible path of consumption, then : $\forall t,\ 0 \leq c_t \leq h_0^{\alpha+\gamma}(1+\lambda)^{(\alpha+\gamma)t}$. This shows that all feasible paths of consumption are compact for this topology. Assumption **H2** ensures that function :

$$U(\mathbf{c}) = \sum_{t=0}^{+\infty}\beta^t u(c_t, \theta_t)$$

is continuous for the product topology. Existence of the solution rise from these results.

6. Appendix 2

It's enough to show that for any initial conditions, $h_0 > 0$, the stationary path $(h_0, h_0, ..., h_0, ...)$ is not optimal. Let $\epsilon > 0$ be a sufficiently small number such as $1 + \lambda\phi(\epsilon) \leq 1 + \lambda$ and a path $\mathbf{h} = (h_0, h_1, ..., h_t, ...)$ which verify $h_t = h_0(1 + \lambda\phi(\epsilon))$, $\forall t \geq 1$. The consumption path associated with this human capital path is $\mathbf{c}_\epsilon = (c_{0_\epsilon}, c_{1_\epsilon}, ..., c_{t_\epsilon}, ...)$ that is : $c_{0_\epsilon} = h_0^{\alpha+\gamma}(1-\epsilon)^\alpha$ and $c_{t_\epsilon} = h_0^{\alpha+\gamma}(1+\lambda\phi(\epsilon))^{\alpha+\gamma}$, $\forall t \geq 1$. Moreover, let $(h_0, h_0, ..., h_0, ...)$ be a human capital path and $\mathbf{c}^*$ be a consumption path which satisfy : $c_t^* = h_0^{\alpha+\gamma}$. Compare the utilities generated by these sequences of consumptions, we have :

$$\Delta_\epsilon = \sum_{t=0}^{+\infty} \beta^t c_{t_\epsilon}^\mu \left(1 - \psi(\frac{h_{t+1}}{h_t})\right)^\zeta - \sum_{t=0}^{+\infty} \beta^t c_t^{*\mu} (1 - \psi(1))^\zeta$$

Since $\psi(1) = 1$, so $\Delta_\epsilon > 0$. All optimal paths of human capital are increasing.

7. Appendix 3

We give the proof of the Proposition 4 in several stages.

1. Let $\mathbf{h}^*$ be an equilibrium. One can show that any equilibrium is increasing, that is $h_{t+1}^* > h_t^*$, $\forall t \geq 0$ (Proceed as in the previous appendix). Moreover, since the utility function verifies the Inada condition, the optimal consumptions are strictly positive on each date. Hence, $h_{t+1}^* < (1+\lambda)h_t^*$, for all t. This ends the first part of the claim. It is easy to show that $\mathbf{h}^*$ verifies the Euler equation (see Le Van & Dana 2003). Let us show now that the transversality condition is satisfied. Let $V_{\mathbf{h}^*}(h_0)$ be the value function of this program, one has :

$$V_{\mathbf{h}^*}(h_0) = \max \sum_{t=0}^{+\infty} \beta^t u(c_t, \theta_t)$$

Under the constraints $(\forall t)$,

$$\begin{aligned} 0 \leq c_t &\leq G(h_t^*) f(\theta_t h_t) \\ h_{t+1} &= h_t(1 + \lambda\phi(1-\theta_t)) \\ 0 \leq \theta_t &\leq 1, \ h_0 > 0 \text{ given} \end{aligned}$$

One can verify that $V_{\mathbf{h}^*}$ is concave and differentiable (Beneviste & Scheinkman 1979) and :

$$\begin{aligned} V'_{\mathbf{h}^*}(h_0) &= \alpha\mu h_0^{(\alpha+\gamma)\mu-1} \left(\psi(\tfrac{h_1^*}{h_0})\right)^{\alpha\mu} \left(1 - \psi(\tfrac{h_1^*}{h_0})\right)^\zeta \\ &-\alpha\mu h_0^{(\alpha+\gamma)\mu-1} \tfrac{h_1^*}{h_0} \psi'(\tfrac{h_1^*}{h_0}) \left(\psi(\tfrac{h_1^*}{h_0})\right)^{\alpha\mu-1} \left(1 - \psi(\tfrac{h_1^*}{h_0})\right)^\zeta \\ &+\zeta h_0^{(\alpha+\gamma)\mu-1} \tfrac{h_1^*}{h_0} \psi'(\tfrac{h_1^*}{h_0}) \left(\psi(\tfrac{h_1^*}{h_0})\right)^{\alpha\mu} \left(1 - \psi(\tfrac{h_1^*}{h_0})\right)^{\zeta-1} \end{aligned}$$

Moreover, since $\mathbf{h}^*$ is a equilibrium, it must verify $0 \leq h_t^* \leq h_0(1+\lambda)^t$ for all t. Consequently, $c_t^* \leq [h_0(1+\lambda)^t]^{\alpha+\gamma}$ and

$$0 \leq V_{\mathbf{h}^*}(h_0) = \sum_{t=0}^{+\infty} \beta^t u(c_t^*, \theta_t^*) \leq h_0^{(\alpha+\gamma)\mu} \sum_{t=0}^{+\infty} [\beta(1+\lambda)^{(\alpha+\gamma)\mu}]^t$$

Like $V'_{\mathbf{h}^*}(0) = 0$, we have for all t :

$$\frac{h_t^{*(\alpha+\gamma)\mu}}{1-\beta(1+\lambda)^{(\alpha+\gamma)\mu}} \geq V_{\mathbf{h}^*} - V_{\mathbf{h}^*}(0) \geq V'_{\mathbf{h}^*}(h_t^*)h_t^*$$

Since,

$$\begin{aligned}
V'_{\mathbf{h}^*}(h_t^*) &= \alpha\mu\, h_t^{*(\alpha+\gamma)\mu-1}\left(\psi(\tfrac{h_{t+1}^*}{h_t^*})\right)^{\alpha\mu}\left(1-\psi(\tfrac{h_{t+1}^*}{h_t^*})\right)^{\zeta} \\
&-\alpha\mu\, h_t^{*(\alpha+\gamma)\mu-1}\tfrac{h_{t+1}^*}{h_t^*}\psi'(h_t^{*(\alpha+\gamma)\mu-1})\left(\psi(\tfrac{h_{t+1}^*}{h_t^*})\right)^{\alpha\mu-1}\left(1-\psi(\tfrac{h_{t+1}^*}{h_t^*})\right)^{\zeta} \\
&+\zeta\, h_t^{*(\alpha+\gamma)\mu-1}\tfrac{h_{t+1}^*}{h_t^*}\psi'(h_t^{*(\alpha+\gamma)\mu-1})\left(\psi(\tfrac{h_{t+1}^*}{h_t^*})\right)^{\alpha\mu}\left(1-\psi(\tfrac{h_{t+1}^*}{h_t^*})\right)^{\zeta-1}
\end{aligned}$$

and $h_t^* \leq h_0(1+\lambda)^t$. Multiply the two previous equations by β^t, we obtain the transversality condition :

$$\begin{aligned}
\lim_{t\to+\infty}\ &\beta^t h_t^{*(\alpha+\gamma)\mu-1}\left(\psi(\tfrac{h_{t+1}^*}{h_t^*})\right)^{\alpha\mu-1}\left(1-\psi(\tfrac{h_{t+1}^*}{h_t^*})\right)^{\zeta-1} \\
&\Big[\alpha\mu\left(1-\psi(\tfrac{h_{t+1}^*}{h_t^*})\right)\left(\psi(\tfrac{h_{t+1}^*}{h_t^*})-\tfrac{h_{t+1}^*}{h_t^*}\psi'(\tfrac{h_{t+1}^*}{h_t^*})\right) \\
&+\zeta\,\tfrac{h_{t+1}^*}{h_t^*}\psi'(\tfrac{h_{t+1}^*}{h_t^*})\psi(\tfrac{h_{t+1}^*}{h_t^*})\left(1-\psi(\tfrac{h_{t+1}^*}{h_t^*})\right)\Big]\, h_t^* = 0
\end{aligned}$$

The Euler condition implies :

$$\begin{aligned}
\lim_{t\to+\infty}\ &\beta^t h_t^{*(\alpha+\gamma)\mu-1}\psi'(\tfrac{h_{t+1}^*}{h_t^*})\left(\psi(\tfrac{h_{t+1}^*}{h_t^*})\right)^{\alpha\mu-1}\left(1-\psi(\tfrac{h_{t+1}^*}{h_t^*})\right)^{\zeta-1} \\
&\left[\alpha\mu\left(1-\psi(\tfrac{h_{t+1}^*}{h_t^*})\right)-\zeta\psi(\tfrac{h_{t+1}^*}{h_t^*})\right] h_{t+1}^* = 0
\end{aligned}$$

2. We prove the converse now. Let $(\mathbf{c}^*,\mathbf{h}^*)$ and $(\mathbf{c},\mathbf{h})$ be two sequences sets with the same initial condition h_0. The last verifies $\forall t,\ 0 \leq c_t \leq (h_t^*)^\gamma \left(h_t\psi(\frac{h_{t+1}}{h_t})\right)^\alpha$. Show that $\sum_{t=0}^T \beta^t u(c_t^*,\theta_t^*) - \sum_{t=0}^T \beta^t u(c_t,\theta_t) \geq 0$. Observe that u, $(x,y) \to x\psi(\frac{y}{x})$ and ψ are concave functions, hence :

$$\begin{aligned}
\Delta_T \geq \sum_{t=0}^T \beta^t &\Big[(h_t^*-h_t)\left(u'_1(c_t^*,\theta_t^*)\frac{\partial c_t}{\partial h_t}(h_t^*,h_{t+1}^*)+u'_2(c_t^*,\theta_t^*)\frac{\partial \theta_t}{\partial h_t}(h_t^*,h_{t+1}^*)\right) \\
&+(h_{t+1}^*-h_{t+1})\left(u'_1(c_t^*,\theta_t^*)\tfrac{\partial c_t}{\partial h_{t+1}}(h_t^*,h_{t+1}^*)+u'_2(c_t^*,\theta_t^*)\tfrac{\partial \theta_t}{\partial h_{t+1}}(h_{t+1}^*-h_{t+1})\right)\Big]
\end{aligned}$$

Where $u'_1(c_t^*,\theta_t^*) = \frac{\partial u}{\partial c_t}(c_t^*,\theta_t^*)$ and $u'_2(c_t^*,\theta_t^*) = \frac{\partial u}{\partial \theta_t}(c_t^*,\theta_t^*)$. By the Euler equation,

$$\begin{aligned}
\Delta_T \geq\ &\beta^T h_T^{*(\alpha+\gamma)\mu-1}\left(\psi(\tfrac{h_{T+1}^*}{h_T^*})\right)^{\alpha\mu-1}\left(1-\psi(\tfrac{h_{T+1}^*}{h_T^*})\right)^{\zeta-1} \\
&\left[\alpha\mu\left(1-\psi(\tfrac{h_{T+1}^*}{h_T^*})\right)-\zeta\psi(\tfrac{h_{T+1}^*}{h_T^*})\right]\psi'(\tfrac{h_{T+1}^*}{h_T^*})h_{T+1}^*
\end{aligned}$$

The transversality condition is written $\lim_{T\to+\infty}\Delta_T = 0$ since $\psi' < 0$.

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mazambatedie@gmail.com
http://mazambatedie.free.fr