A Simple Version of the Lucas Model

Mazamba Tédie

Abstract

This discrete-time version of the Lucas model do not include the physical capital. We integrate in the utility function the leisure time. We examine the social planner and the competitive equilibrium. The main conclusions are that the consumer always chooses to train, the human capital growth rate increases with the externality and the quality of training, and that the equilibrium defined by Lucas (1988) is a competitive equilibrium under some conditions.

1. Introduction

This model is a discrete-time version of the model of Lucas without physical capital. The consumer devotes the fraction $\theta$ of his non-leisure time to current production and the remaining $(1-\theta)$ to human capital accumulation. We consider that the utility of consumer increases with his leisure time. This assumption implies that the utility increases with the human capital accumulation that is with the training. Following Lucas (1988), the human capital has: 1- an external effect through the externality. 2- an internal effect which increases the productivity through the medium of training.

This paper is organized into seven sections. Section 2 introduces assumptions and examines the social planner problem. After defining the equilibrium (according to Lucas and Romer) and competitive equilibrium, section 3 shows that an equilibrium is a competitive equilibrium. The following sections conclude and give some proofs.

2. Social Planer

The utility function is concave ($0 < \mu < 1$ and $0 < \zeta < 1$):

$$\max_{\phi} \sum_{t=0}^{+\infty} \beta^t c_t^\mu (1-\theta_t)^\zeta$$

Subject to,

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\[ \forall t \geq 0, \quad 0 \leq c_t \leq h_t^\gamma (\theta_t h_t)\alpha \]
\[ h_{t+1} = h_t (1 + \lambda \phi(1 - \theta_t)) \]
\[ 0 < \alpha < 1, \quad \gamma \geq 0, \quad 0 \leq \theta_t \leq 1, \quad h_0 > 0 \] given

We make the following assumptions :

**H1** : \( \phi \) is concave, increasing and twice continuously differentiable. \( \phi(0) = 0, \phi(1) = 1, \lambda > 0 \) and \( \phi'(0) > 1 \).

**H2** : \( 0 < \beta(1 + \lambda)^{(\alpha + \gamma)\mu} < 1 \).

The parameter \( \lambda \) balanced the technology of training. Let us define the function \( \psi : [1, 1 + \lambda] \to [0, 1] \) by \( \psi(x) = 1 - \phi^{-1} \left( \frac{1}{\lambda}(x - 1) \right) \). Where \( \phi^{-1} \) denotes the inverse function of \( \phi \). \( \psi \) is clearly decreasing. It is easy to verify that : \( \psi(1) = 1 \) and \( \psi(1 + \lambda) = 0 \). This function gives the working time when the human capital grows by factor \( x \). \( \psi \) is continuously differentiable, decreasing, with \( \psi'(1) = -\frac{1}{\lambda \phi'(0)}, \psi'(1 + \lambda) = -\frac{1}{\lambda \phi'(1)} \) and concave.

The problem becomes :
\[
\max_{t=0}^{+\infty} \sum_{t=0}^{+\infty} \beta^t h_t^{(\alpha + \gamma)\mu} \left( \psi \left( \frac{h_{t+1}}{h_t} \right) \right)^\alpha \mu \left( 1 - \psi \left( \frac{h_{t+1}}{h_t} \right) \right)^\zeta
\]
Subject to :
\[ \forall t \geq 0, \quad h_t \leq h_{t+1} \leq h_t (1 + \lambda), \quad \text{and} \quad h_0 > 0 \] given.

**Proposition 1** Under **H1-H2**, there exists a solution.

**Proof.** See the appendix 1.

**Proposition 2** Each optimal path of human capital \( h = (h_0, h_1, \ldots, h_t, \ldots) \) verifies \( h_0 < h_1 < \cdots < h_t < \cdots \).

**Proof.** See the appendix 2.

**Proposition 3** Under assumptions **H1,H2** and **H3** : \( (\alpha + \gamma)\mu - 1 < 0 \) :

(a) The optimal path of human capital has a constant growth rate, strictly positive and which increases with parameter \( \gamma \).

(b) The optimal path of human capital is an increasing function of \( \lambda \).

**Proof.** We prove (a) in several stages.

1. Let \( V \) be the **Value Function** of our problem of optimal growth :
\[
V(h_0) = \max_{t=0}^{+\infty} \sum_{t=0}^{+\infty} \beta^t h_t^{(\alpha + \gamma)\mu} \left( \psi \left( \frac{h_{t+1}}{h_t} \right) \right)^\alpha \mu \left( 1 - \psi \left( \frac{h_{t+1}}{h_t} \right) \right)^\zeta
\]
Under the constraints : \( \forall t \geq 0, h_t \leq h_{t+1} \leq h_t (1 + \lambda), \) and \( h_0 > 0 \) given.
This value function satisfied (see Le Van & Morhaim 2002) :

2.
$$V(h_0) = A h_0^{(\alpha + \gamma)\mu}$$

Let us consider $h_0$, the optimal value $h_1$ of the human capital at date 1 is the solution of the following equation:

$$h_0^{(\alpha + \gamma)\mu} \max_{y \in [h_0, (1 + \lambda) h_0]} \left\{ \left( \psi \left( \frac{y}{h_0} \right) \right)^{\alpha \mu} \left( 1 - \psi \left( \frac{y}{h_0} \right) \right)^{\zeta} + \beta A \left( \frac{y}{h_0} \right)^{(\alpha + \gamma)\mu} \right\}$$

We can see that $h_1^* = \nu h_0$ where $\nu$ is the solution of the equation:

$$\max \left\{ \left( \psi(z) \right)^{\alpha \mu} (1 - \psi(z))^\zeta + \beta A(\gamma) z^{(\alpha + \gamma)\mu} \right\}.$$ Since the problem is stationary, if $\{h_t\}$ is the optimal path, then $h_t = \nu^t h_0, \forall t$.

2. We know that the human capital path verifies $h_{t+1} > h_t, \forall t \geq 0$. The Euler equation is given by:

$$h_t^{(\alpha + \gamma)\mu - 1} \psi'(\frac{h_{t+1}}{h_t}) \Psi\left( \frac{h_{t+1}}{h_t} \right) \left[ \alpha \mu \left( 1 - \psi\left( \frac{h_{t+1}}{h_t} \right) \right) - \zeta \psi'\left( \frac{h_{t+1}}{h_t} \right) \right]$$

$$= -\beta (\alpha + \gamma) h_t^{(\alpha + \gamma)\mu - 1} \Psi\left( \frac{h_{t+1}}{h_t} \right) \left[ \psi\left( \frac{h_{t+1}}{h_t} \right) \right] \right] \left( 1 - \psi\left( \frac{h_{t+1}}{h_t} \right) \right) - \zeta \psi'\left( \frac{h_{t+1}}{h_t} \right) \right]$$

$$+ \beta h_{t+1} \psi'\left( \frac{h_{t+1}}{h_t} \right) \Psi\left( \frac{h_{t+1}}{h_t} \right) \left[ \alpha \mu \left( 1 - \psi\left( \frac{h_{t+1}}{h_t} \right) \right) - \zeta \psi'\left( \frac{h_{t+1}}{h_t} \right) \right]$$

With $\Psi\left( \frac{h_{t+1}}{h_t} \right) = \left( \psi\left( \frac{h_{t+1}}{h_t} \right) \right)^{\alpha \mu - 1} \left( 1 - \psi\left( \frac{h_{t+1}}{h_t} \right) \right)^{\zeta - 1}$. This equation gives the human capital growth rate that is constant ($\nu$):

$$1 = \frac{\lambda}{\alpha} \frac{\nu}{\psi(\nu)} \left( 1 - \beta \nu^{(\alpha + \gamma)\mu} \right) - \frac{\beta}{\alpha} (\alpha + \gamma) \nu^{(\alpha + \gamma)\mu - 1} \frac{\psi(\nu)}{\psi'(\nu)} + \beta \nu^{(\alpha + \gamma)\mu}$$

Let $F(\nu) = \frac{\ell}{\alpha} \frac{\nu}{\psi(\nu)} \left( 1 - \beta \nu^{(\alpha + \gamma)\mu} \right)$ with $G(\nu) = -\frac{\beta}{\alpha} (\alpha + \gamma) \nu^{(\alpha + \gamma)\mu} \frac{\psi(\nu)}{\psi'(\nu)} + \beta \nu^{(\alpha + \gamma)\mu}$.

Functions $F$ and $G$ are decreasing since:

$$F'(x) = (1 - \beta x^{(\alpha + \gamma)\mu}) \frac{\ell}{\alpha} \frac{\psi'(x)}{\psi(x)} = \frac{\beta \nu^{(\alpha + \gamma)\mu - 1} \psi(x)}{\psi'(x)} x^{(\alpha + \gamma)\mu - 1} < 0, \quad G'(x) = -\frac{\beta}{\alpha} (\alpha + \gamma) x^{(\alpha + \gamma)\mu} - \frac{\beta \nu^{(\alpha + \gamma)\mu}}{\psi'(x)} x < 0.$$ Moreover, $F(1 + \lambda) = 0$, $\lim_{x \to 1} F(x) = +\infty$, $G(1) = \frac{\beta (\alpha + \gamma)}{\alpha} \lambda \nu(0)$ and $G(1 + \lambda) = \beta (1 + \lambda)^{(\alpha + \gamma)\mu} < 1$ according to H2. Hence, there exists a unique solution $\nu \in [1, 1 + \lambda]$.

3. We know that the value function verifies the Bellman equation:

$$V(h) = h^{(\alpha + \gamma)\mu} \max_{\nu \in [1, 1 + \lambda]} \left\{ \left( \psi(\nu) \right)^{\alpha \mu} (1 - \psi(\nu))^\zeta + \beta A(\gamma) \nu^{(\alpha + \gamma)\mu} \right\}$$

The derivative of function $(\psi(\nu))^{\alpha \mu} (1 - \psi(\nu))^\zeta + \beta A(\gamma) \nu^{(\alpha + \gamma)\mu}$ is cancelled:

$$-\alpha \mu \psi'(\nu^*) (\psi(\nu^*))^\zeta + \zeta (\nu^*) (\psi(\nu^*))^{\alpha \mu} (1 - \psi(\nu^*))^{\zeta - 1} = \beta A(\gamma) (\alpha + \gamma) \nu^{(\alpha + \gamma)\mu - 1}$$

When $\gamma$ increases, the graph of the function $\beta A(\gamma) (\alpha + \gamma) \nu^{(\alpha + \gamma)\mu - 1}$ moves to the top while the left-hand side remains constant. Consequently, the growth rate increases with the parameter of the externality. This ends the proof of the claim (a).

4. Let us rewrite the Euler equation: $1 = F(\lambda) + G(\lambda)$. Note that $\lambda < \lambda' \Rightarrow \psi_\lambda < \psi_{\lambda'}$ and $-\psi_\lambda' < -\psi_{\lambda'}$. Hence, $F$ and $G$ are increasing with $\lambda$. Moreover, $F$ and $G$ are decreasing with $\nu$, then:

$$\frac{d\nu}{d\lambda} = -\left[ \frac{\partial F}{\partial \lambda} + \frac{\partial G}{\partial \lambda} \right] \frac{\partial G}{\partial \nu} \frac{\partial F}{\partial \nu} > 0$$

3
3. Equilibrium and Competitive Equilibrium

We introduce the concepts of equilibrium (according to Lucas and Romer) and competitive equilibrium. Take a human capital path \( \hat{h} = (\hat{h}_1, ..., \hat{h}_t, ...) \) to be given. Given \( \hat{h} \), consider the problem:

\[
\max_{c_t} \sum_{t=0}^{+\infty} \beta^t u(c_t, \theta_t)
\]

Under the constraints,

\[
\forall t, \quad 0 \leq c_t \leq G(\hat{h}) f(\theta_t h_t) \\
h_{t+1} = h_t (1 + \lambda \phi (1 - \theta_t)) \\
0 \leq \theta_t \leq 1, \quad h_0 > 0 \text{ given}
\]

The solution \( h = (h_0, h_1, ..., h_t, ...) \) of this model depends on \( \hat{h} \). In others words, \( h = \Phi(\hat{h}) \). A equilibrium is a human capital path \( h^* = (h_0^*, ..., h_t^*, ...) \) such that \( h^* = \Phi(h^*) \).

In order to define a competitive equilibrium, we need before to define the space of the prices which supports this equilibrium. Observe that all feasible paths of consumption \( c_t \) verify for all \( t \):

\[
0 \leq c_t \leq h_t^{\alpha+\gamma} + \gamma_t \text{ with } h_t \leq h_0 (1 + \lambda)^t.
\]

In others words, \( c_t \) belongs to:

\[
\ell^\infty = \left\{ c : \sup_{t=0,..,+\infty} \left| c_t \right| \left(1 + \lambda\right)^{\alpha+\gamma} t < +\infty \right\}
\]

Let \( \ell^\infty \) be the set of non negative sequences of \( \ell^\infty \). The price sequence \( p_t \) is such as all consumption paths \( c_t \) verify \( \sum_{t=0}^{+\infty} p_t c_t < +\infty \). Likewise, the wage path \( w_t \) is such as \( \sum_{t=0}^{+\infty} w_t h_t < +\infty \). In order to satisfy these two conditions, we must take the prices space and the wages space as follows:

\[
\ell^1_p = \left\{ p : \sum_{t=0}^{+\infty} |p_t|(1 + \lambda)^{\alpha+\gamma} t < +\infty \right\} ; \ell^1_w = \left\{ w : \sum_{t=0}^{+\infty} |w_t|(1 + \lambda)^t < +\infty \right\}
\]

Let us denote \( \ell^1_+ \), the set of non-negative sequences of \( \ell^1 \).

We define a competitive equilibrium for the model of Lucas.

A collection of sequences \( (h^*, c^*, \theta^*, p^*, w^*) \) is a competitive equilibrium if:

1. \( (c^*, \theta^*) \) is a solution of the consumer program:

\[
\max_{c_t, \theta_t} \sum_{t=0}^{+\infty} \beta^t u(c_t, \theta_t)
\]

Under the constraints,

\[
\sum_{t=0}^{+\infty} p^*_t c_t \leq \sum_{t=0}^{+\infty} w^*_t \theta_t h_t + \Pi^* \\
\forall t \geq 0, \quad \theta_t = \psi\left(\frac{h_{t+1}}{h_t}\right), \quad h_0 > 0 \text{ given}
\]

2. \( \theta^* \) is a solution of the firm program:
Proposition 5

Under the assumptions of proposition 3 and \( \nu \) according to Euler equation, this rate capital sequence that increases at constant rate and satisfies the Euler equation. Indeed, let us show that exists a human

Proof.

1. We know that if \( \psi \) is a competitive equilibrium.

2. Euler equation \( (\forall t \geq 0) \),

\[
\alpha \mu h_t^{*(\alpha+\gamma)\mu-1} \psi' \left( \frac{h_{t+1}^*}{h_t^*} \right) \left( \psi \left( \frac{h_{t+1}^*}{h_t^*} \right) \right)^{\alpha \mu - 1} \left( 1 - \psi \left( \frac{h_{t+1}^*}{h_t^*} \right) \right)^\zeta \\
- \zeta h_t^{*(\alpha+\gamma)\mu-1} \psi' \left( \frac{h_{t+1}^*}{h_t^*} \right) \left( \psi \left( \frac{h_{t+1}^*}{h_t^*} \right) \right)^{\alpha \mu} \left( 1 - \psi \left( \frac{h_{t+1}^*}{h_t^*} \right) \right)^{-1} \\
+ \beta \alpha \mu h_t^{*(\alpha+\gamma)\mu-1} \left( \psi \left( \frac{h_{t+1}^*}{h_t^*} \right) \right)^{\alpha \mu} \left( 1 - \psi \left( \frac{h_{t+1}^*}{h_t^*} \right) \right)^\zeta \\
- \beta \alpha \mu h_t^{*(\alpha+\gamma)\mu-1} \left( \psi \left( \frac{h_{t+1}^*}{h_t^*} \right) \right)^{\alpha \mu} \left( 1 - \psi \left( \frac{h_{t+1}^*}{h_t^*} \right) \right)^{-1} \\
+ \beta \zeta h_t^{*(\alpha+\gamma)\mu-1} \left( \psi \left( \frac{h_{t+1}^*}{h_t^*} \right) \right)^{\alpha \mu} \left( 1 - \psi \left( \frac{h_{t+1}^*}{h_t^*} \right) \right)^{-1} = 0
\]

3. Transversality condition,

\[
\lim_{t \to +\infty} \beta^t h_t^{*(\alpha+\gamma)\mu-1} \psi' \left( \frac{h_{t+1}^*}{h_t^*} \right) \left( \psi \left( \frac{h_{t+1}^*}{h_t^*} \right) \right)^{\alpha \mu - 1} \left( 1 - \psi \left( \frac{h_{t+1}^*}{h_t^*} \right) \right)^\zeta \\
- \zeta \psi' \left( \frac{h_{t+1}^*}{h_t^*} \right) \left( \psi \left( \frac{h_{t+1}^*}{h_t^*} \right) \right)^{\alpha \mu - 1} h_{t+1}^* = 0
\]

Proof. See the appendix 3.

Proposition 4 \( h^* \) is a equilibrium from \( h_0 > 0 \) if and only if it verifies the three following conditions:

1. Interiority : \( \forall t \geq 0, h_t^* < h_{t+1}^* < (1 + \lambda)h_t^*, h_0^* = h_0 > 0 \)

2. Euler equation \( (\forall t \geq 0) \),

\[
\Pi^* = \max_g \left\{ \sum_{t=0}^{+\infty} \beta_t (h_t^*)^\gamma (\theta_t h_t^*)^\alpha - \sum_{t=0}^{+\infty} w_t \theta_t h_t^* \right\}
\]

3. Equilibrium on the goods and services market :

\[
\forall t \geq 0, c_t^* = (h_t^*)^\gamma (\theta_t^* h_t^*)^\alpha
\]

Proof. We know that if \( h^* \) is an equilibrium then it verifies interiority, the Euler equation and the transversality condition. We can associate with this equilibrium the stationary sequence \( \theta^* = (\psi(\nu)) \), a consumption sequence \( c^* \), a price system \( \pi^* \), wage \( w^* \) such as the collection of sequences \( (h^*, c^*, \theta^*, \pi^*, w^*) \) is a competitive equilibrium.

Proof. 1. We know that if \( h^* \) is an equilibrium then it verifies interiority, the Euler equation and the transversality condition. In addition, let us show that exists a human capital sequence that increases at constant rate and satisfies the Euler equation. Indeed, according to Euler equation, this rate \( \nu \) must satisfy :

\[
1 = \zeta \psi(\nu) \left( 1 - \beta V(\alpha+\gamma) \right) - \beta V(\alpha+\gamma) (1 - \psi(\nu)) \psi(\nu) + \beta V(\alpha+\gamma) \equiv V(\nu)
\]

5
Let $F(\nu) = \frac{\psi(\nu)}{\alpha \mu} \left(1 - \beta^{(\alpha + \gamma)\mu}\right)$ and $H(\nu) = -\beta^{(\alpha + \gamma)\mu} \frac{\psi(\nu)}{\psi(\nu)} + \beta^{(\alpha + \gamma)\mu}$. We know that $F$ is decreasing, $\lim_{x \to -1} F(x) = +\infty$ and that $F(1 + \lambda) = 0$. We show that $H$ is also decreasing : 

$$H'(\nu) = -\beta^{(\alpha + \gamma)\mu} \left[\left((\alpha + \gamma)\mu - 1\right) \left(\frac{\psi(\nu)}{\psi(\nu)} - \nu\right) - \nu^2 \frac{\psi(\nu) \psi'(\nu)}{(\psi(\nu))^2}\right] < 0.$$ 

One has $V(x) = F(x) + H(x)$, $V'(x) = F'(x) + H'(x)$, $\lim_{x \to -1} V(x) = \lim_{x \to 1} V(x)$ and $H(x) = +\infty$ and $V(1 + \lambda) = F(1 + \lambda) + H(1 + \lambda) = \beta(1 + \lambda)^{(\alpha + \gamma)\mu} < 1$ according to H2.

Consequently, there exists a unique solution $\nu$ which belongs to $]1, 1 + \lambda[$. It’s easy to show that this rate is weaker than the rate of social planer program which is the solution of the equation $1 = F(\nu) + G(\nu)$, since $G(\nu) = H(\nu) - \frac{\psi(\nu)}{\psi(\nu)} \beta^{(\alpha + \gamma)\mu - 1}$. Let $h^*$ be the trajectory defined by $h_0^* = h_0$, $h_{t+1}^* = \nu h_t^*$, $\forall t$. Obviously, it satisfies the interiority and Euler equation. We must show that it verifies the transversality condition to conclude that $h^*$ is an equilibrium. Now,

$$\begin{align*}
\beta h_t^* \psi(h_t^*) \frac{\psi(h_{t+1}^*)}{\psi(h_t^*)} \left(1 - \psi(h_{t+1}^*)\right) & - \psi(h_t^*) \left(1 - \psi(h_{t+1}^*)\right) A(h_{t+1}^*) \\
= & \beta h_t^* \psi(h_t^*) \mu^{(\alpha + \gamma)\mu} \left(1 - \psi(h_t^*)\right) A(h_{t+1}^*) \\
\leq & h_0^* \left(\psi(h_t^*) \mu^{(\alpha + \gamma)\mu - 1} \left(1 - \psi(h_t^*)\right) \right) \left(1 - A(h_{t+1}^*)\right) \left[\beta(1 + \lambda)^{(\alpha + \gamma)\mu}\right]^t
\end{align*}$$

Where $A' = \alpha \mu \left(1 - \psi(h_{t+1}^*)\right) - \psi(h_t^*)$. Assumption H2 implies :

$$\lim_{t \to +\infty} \beta h_t^* \psi(h_t^*) \frac{\psi(h_{t+1}^*)}{\psi(h_t^*)} \left(1 - \psi(h_{t+1}^*)\right) A' = 0$$

This is the transversality condition.

2. We show that this trajectory is a competitive equilibrium. Let us define the price path and the wage path, $p^*$, $w^*$ by :

$$\begin{align*}
p_t^* & = \beta \mu \frac{\partial u(c_t^*)}{\partial \theta_t} = \mu \beta h_t^* \left(\psi(h_t^*) \mu^{(\alpha + \gamma)\mu - 1} \left(1 - \psi(h_t^*)\right)\right) \\
w_t^* & = \beta h_t^* \psi(h_t^*) \mu^{(\alpha + \gamma)\mu - 1} \left(1 - \psi(h_t^*)\right) \left[\alpha \mu (1 - \psi(h_t^*) - \psi(h_t^*))\right]
\end{align*}$$

Where $h_t^* = \nu t h_0^*$.

a) It is easy to see that the sequence $\theta^*$ defined by $\theta_t^* = \psi(h_t)$, for all $t$, maximizes the profit of the enterprise according to $p^*$ and $w^*$.

b) In order to prove that the consumption path and the working time path $(c_t^*, \theta_t^*)$ maximize the consumer utility, consider :

$$\Delta T = \sum_{t=0}^{T} \beta^t u(c_t^*, \theta_t^*) - \sum_{t=0}^{T} \beta^t u(c_t, \theta_t)$$

Since $\sum_{t=0}^{+\infty} \beta^t u(c_t^*) = \sum_{t=0}^{+\infty} w_t^* \theta_t h_t^* + \Pi^*$ and $\sum_{t=0}^{+\infty} \beta^t u(c_t^*) c_t < \sum_{t=0}^{+\infty} w_t^* \theta_t h_t + \Pi^*$ with $\theta_t = \psi(h_t^*)$, one has :

$$\Delta T \geq \sum_{t=0}^{T} \beta^t \left[\left(h_t - h_t^*\right) \left(\alpha \mu h_t^{(\alpha + \gamma)\mu - 1} \left(1 - \psi(h_{t+1}^*)\right) \Phi(h_{t+1}^*)\right) + \frac{\psi(h_{t+1}^*)}{\psi(h_t^*)} \left(1 - \psi(h_{t+1}^*)\right) \xi^-(h_{t+1}^*)\right]$$

$$+ \frac{\psi(h_{t+1}^*)}{\psi(h_t^*)} \left(1 - \psi(h_{t+1}^*)\right) \xi^-(h_{t+1}^*)\right]$$

$$- \frac{\psi(h_{t+1}^*)}{\psi(h_t^*)} \left(1 - \psi(h_{t+1}^*)\right) \xi^-(h_{t+1}^*)\right]$$
According to H4 has:

d) To complete this proof, let us show that is continuous for the product topology. Existence of the solution rise from these results.

The goods market is balanced since for all \( t \):

\[
\sum_{t=0}^{+\infty} p_t (1 + \lambda)^{(\alpha+\gamma)t} < \mu B' h_0^{\alpha+\gamma}(\mu - 1) \sum_{t=0}^{+\infty} [\beta (1 + \lambda)^{(\alpha+\gamma)t}]^t < +\infty
\]

According to H2 and with \( B' = (\psi(\nu))^{\alpha(\mu-1)} (1 - \psi(\nu))^\xi \). Likewise,

\[
\sum_{t=0}^{+\infty} w_t (1 + \lambda)^t = C' h_0^{\alpha+\gamma}(\mu - 1) \sum_{t=0}^{+\infty} [\beta (1 + \lambda)^{(\alpha+\gamma)\mu-1}]^t < +\infty
\]

According to H4, \( 1 < \nu < 1 + \lambda \) and where \( C' = (\psi(\nu))^{\alpha(\mu-1)} (1 - \psi(\nu))^\xi - \zeta \psi(\nu) \). This ends the proof. A collection of paths \( (h^*, e^*, \theta^*, p^*, w^*) \) is a competitive equilibrium.

4. Conclusion

This discrete-time version of the Lucas model solves the social planner program and shows that an equilibrium for this model is a competitive equilibrium. Moreover, the model concludes that: 1- when the utility depends on consumption and leisure time, the consumer always prefers to increase his skill level. 2- the quality of training increases the human capital growth rate. 3- the externality is related positively to the human capital growth rate through it contribution to the productivity of all factors of production.

5. Appendix 1

It’s easy to verify that if \( c = (c_0, c_1, ..., c_t, ...) \) is a feasible path of consumption, then:

\[
\forall t, 0 \leq c_t \leq h_0^{\alpha+\gamma}(1 + \lambda)^{(\alpha+\gamma)t}
\]

This shows that all feasible paths of consumption are compact for this topology. Assumption H2 ensures that function:

\[
U(c) = \sum_{t=0}^{+\infty} \beta^t u(c_t, \theta_t)
\]

is continuous for the product topology. Existence of the solution rise from these results.
6. Appendix 2

It’s enough to show that for any initial conditions, \( h_0 > 0 \), the stationary path \((h_0, h_0, \ldots, h_0, \ldots)\) is not optimal. Let \( \epsilon > 0 \) be a sufficiently small number such as \( 1 + \lambda \phi(\epsilon) \leq 1 + \lambda \) and a path \( h = (h_0, h_1, \ldots, h_t, \ldots) \) which verify \( h_t = h_0 (1 + \lambda \phi(\epsilon)), \forall t \geq 1 \). The consumption path associated with this human capital path is \( c_\epsilon = (c_0, c_1, \ldots, c_t, \ldots) \) that is : \( c_0 = h_0^{\alpha+\gamma}(1 - \epsilon)^\alpha \) and \( c_t = h_0^{\alpha+\gamma}(1 + \lambda \phi(\epsilon))^{\alpha+\gamma}, \forall t \geq 1 \). Moreover, let \((h_0, h_0, \ldots, h_0, \ldots)\) be a human capital path and \( c^*_t = h_0^{\alpha+\gamma} \). Compare the utilities generated by these sequences of consumptions, we have:

\[
\Delta_c = \sum_{t=0}^{+\infty} \beta^t c^*_t \left( 1 - \psi \left( \frac{h_{t+1}}{h_t} \right) \right) - \sum_{t=0}^{+\infty} \beta^t c_\epsilon(t) (1 - \psi(1)) c
\]

Since \( \psi(1) = 1 \), so \( \Delta_c > 0 \). All optimal paths of human capital are increasing.

7. Appendix 3

We give the proof of the Proposition 4 in several stages.

1. Let \( h^* \) be an equilibrium. One can show that any equilibrium is increasing, that is \( h_{t+1}^* > h_t^* \), \( \forall t \geq 0 \) (Proceed as in the previous appendix). Moreover, since the utility function verifies the Inada condition, the optimal consumptions are strictly positive on each date. Hence, \( h_{t+1}^* < (1 + \lambda) h_t^* \) for all \( t \). This ends the first part of the claim. It is easy to show that \( h^* \) verifies the Euler equation (see Le Van & Dana 2003). Let us show now that the transversality condition is satisfied. Let \( V_{h^*}(h_0) \) be the value function of this program, one has:

\[
V_{h^*}(h_0) = \max \sum_{t=0}^{+\infty} \beta^t u(c_t, \theta_t)
\]

Under the constraints (\( \forall t \)),

\[
0 \leq c_t \leq G(h^*_t) f(\theta_t h_t)
\]

\[
h_{t+1} = h_t (1 + \lambda \phi(1 - \theta_t))
\]

\[
0 \leq \theta_t \leq 1, \ h_0 > 0 \ given
\]

One can verify that \( V_{h^*} \) is concave and differentiable (Beneviste & Scheinkman 1979) and :

\[
V'_{h^*}(h_0) = \alpha \mu h_0^{(\alpha+\gamma)\mu-1} \left( \psi \left( \frac{h^*_t}{h_0} \right) \right)^{\alpha\mu} \left( 1 - \psi \left( \frac{h^*_t}{h_0} \right) \right)^{\gamma}
\]

\[
- \alpha \mu h_0^{(\alpha+\gamma)\mu-1} h_t^* \psi' \left( \frac{h^*_t}{h_0} \right) \left( \psi \left( \frac{h^*_t}{h_0} \right) \right)^{\alpha\mu-1} \left( 1 - \psi \left( \frac{h^*_t}{h_0} \right) \right)^{\gamma}
\]

\[
+ \xi h_0^{(\alpha+\gamma)\mu-1} h_t^* \psi' \left( \frac{h^*_t}{h_0} \right) \left( \psi \left( \frac{h^*_t}{h_0} \right) \right)^{\alpha\mu} \left( 1 - \psi \left( \frac{h^*_t}{h_0} \right) \right)^{\gamma-1}
\]

Moreover, since \( h^* \) is a equilibrium, it must verify \( 0 \leq h_t^* \leq h_0 (1 + \lambda)^t \) for all \( t \). Consequently, \( c_t^* \leq [h_0 (1 + \lambda)^t]^{\alpha+\gamma} \) and

\[
0 \leq V_{h^*}(h_0) = \sum_{t=0}^{+\infty} \beta^t u(c_t^*, \theta_t^*) \leq h_0^{(\alpha+\gamma)\mu} \sum_{t=0}^{+\infty} [\beta (1 + \lambda)^t]^{(\alpha+\gamma)\mu} \]

8
Like $V^{'}_{h^*}(0) = 0$, we have for all $t$ :

$$\frac{h_t^{*}h^*_{(a+\gamma)^{\mu}}}{1 - \beta(1 + \lambda)(a+\gamma)^{\mu}} \geq V^{'}_{h^*} - V^{'}_{h^*}(0) \geq V^{'}_{h^*}(h_t^*)h_t^*$$

Since,

$$V^{'}_{h^*}(h_t^*) = \alpha \mu h_t^* h^{(a+\gamma)^{\mu - 1}} (\psi(\frac{h_t^{t+1}}{h_t^*}))^\alpha \mu (1 - \psi(\frac{h_t^{t+1}}{h_t^*}))^\xi$$

$$- \alpha \mu h_t^* h^{(a+\gamma)^{\mu - 1}} h_t^{t+1}(\psi(\frac{h_t^{t+1}}{h_t^*}))^\alpha \mu - 1 (1 - \psi(\frac{h_t^{t+1}}{h_t^*}))^\xi$$

$$+ \zeta h_t^* h^{(a+\gamma)^{\mu - 1}} h_t^{t+1} \psi'(h_t^*(a+\gamma)^{\mu - 1}) (\psi(\frac{h_t^{t+1}}{h_t^*}))^\alpha \mu (1 - \psi(\frac{h_t^{t+1}}{h_t^*}))^{\xi - 1}$$

and $h_t^* \leq h_0(1 + \lambda)^t$. Multiply the two previous equations by $\beta^t$, we obtain the transversality condition :

$$\lim_{t \to +\infty} \beta^t h_t^* (a+\gamma)^{\mu - 1} (\psi(\frac{h_t^{t+1}}{h_t^*}))^\alpha \mu - 1 (1 - \psi(\frac{h_t^{t+1}}{h_t^*}))^\xi - 1$$

$$\alpha \mu \left(1 - \psi(\frac{h_t^{t+1}}{h_t^*})\right) \left(\psi\left(\frac{h_t^{t+1}}{h_t^*}\right) - \frac{h_t^{t+1}}{h_t^*} \psi'(\frac{h_t^{t+1}}{h_t^*})\right)$$

$$+ \zeta h_t^* \psi'(\frac{h_t^{t+1}}{h_t^*}) (1 - \psi(\frac{h_t^{t+1}}{h_t^*}))^\xi = 0$$

The Euler condition implies :

$$\lim_{t \to +\infty} \beta^t h_t^* (a+\gamma)^{\mu - 1} (\psi(\frac{h_t^{t+1}}{h_t^*}))^\alpha \mu - 1 (1 - \psi(\frac{h_t^{t+1}}{h_t^*}))^\xi - 1$$

$$\alpha \mu \left(1 - \psi(\frac{h_t^{t+1}}{h_t^*})\right) - \zeta \psi'(\frac{h_t^{t+1}}{h_t^*}) h_t^{* t+1} = 0$$

2. We prove the converse now. Let $(c^*, h^*)$ and $(c, h)$ be two sequences sets with the same initial condition $h_0$. The last verifies $\forall t, 0 \leq c_t \leq \left(h_t^*(a+\gamma)^t\right)^\alpha$. Show that $\sum_{t=0}^T \beta^t u(c_t^*, h_t^*) - \sum_{t=0}^T \beta^t u(c_t, h_t) \geq 0$. Observe that $u, (x, y) \to x \psi(\frac{y}{x})$ and $\psi$ are concave functions, hence :

$$\Delta_T \geq \sum_{t=0}^T \left(\frac{h_t^* - h_t}{h_t} \left(u_1(c_t^*, h_t^*) \frac{\partial c_t}{\partial h_t}(h_t^*, h_t^{t+1}) + u_2(c_t^*, h_t^*) \frac{\partial h_t}{\partial h_t}(h_t^*, h_t^{t+1})\right)\right)$$

$$+ (h_t^{t+1} - h_t + 1) \left(u_1(c_t^*, h_t^*) \frac{\partial c_t}{\partial h_t}(h_t^*, h_t^{t+1}) + u_2(c_t^*, h_t^*) \frac{\partial h_t}{\partial h_t}(h_t^{t+1} - h_t + 1)\right)$$

Where $u_1(c_t^*, h_t^*) = \frac{\partial u(c_t^*, h_t^*)}{\partial c_t}$ and $u_2(c_t^*, h_t^*) = \frac{\partial u(c_t^*, h_t^*)}{\partial h_t}$. By the Euler equation,

$$\Delta_T \geq \beta^T h_T^* (a+\gamma)^{\mu - 1} (\psi(\frac{h_T^{t+1}}{h_T^*}))^\alpha \mu - 1 (1 - \psi(\frac{h_T^{t+1}}{h_T^*}))^\xi - 1$$

$$\alpha \mu \left(1 - \psi(\frac{h_T^{t+1}}{h_T^*})\right) - \zeta \psi'(\frac{h_T^{t+1}}{h_T^*}) h_T^{* t+1}$$

The transversality condition is written $\lim_{t \to +\infty} \Delta_T = 0$ since $\psi' < 0$. 

9
References


mazambatedie@gmail.com
http://mazambatedie.free.fr